Math 286

Differential Equations

Section 1.1 Handout

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**Population model**. With a basic populaton model, we make the following assumption.

- The growth rate of the population is proportional to the current population.

Letting $t$ be time, the independent variable, $P(t)$ be the population as a function of time, the dependent variable, and $k$ some constant, the above statement can be written as the following equation

$$\frac{dP}{dt} = kP$$

If we were to look for some type of solution to this equation, we would be searching for a function whose derivative is a constant multiple of itself. It is fairly easy to notice here that $P(t) = e^{kt}$ would be a solution for this equation.

$$\frac{dP}{dt} = \frac{d}{dt}e^{kt} = ke^{kt} = kP$$

**Logistic population model**. We will now add to the previous example by considering a population that has a carrying capacity, $N$, which is a maximal population based upon restrictions to population growth such as limited resources. We must now add an additional assumption.

- For small populations, the growth rate is proportional to the current population.
- For populations larger than the carrying capacity, the growth rate is negative.

The expression

$$\left(1 - \frac{P}{N}\right)$$

is nearly equal to one if $P$ is small, and negative if $P > N$, and hence we can find a model fitting the above assumptions as

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P$$

**Population models with harvesting**. Say our model is for a population of trout in a river, and a certain fixed number of trout are removed from the river each year. Other than this **harvesting**, the system is a basic population model.

- The growth rate of the population is proportional to the current population.
- A fixed number $F$ are removed each year.

In this case, the slight modification is

$$\frac{dP}{dt} = kP - F$$

We can also add a harvesting term to the logistic population model from the previous example.

- For small populations, the growth rate is proportional to the current population.
- For populations larger than the carrying capacity, the growth rate is negative.
- A fixed number $F$ are removed each year.

In this case we have

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P - F$$

**Amortization**. In this example we find an equation for the amount of money remaining on a mortgage. Let $B(t)$ be the balance owed on the mortgage $t$ years after it is taken out, $r$ be the annual interest rate, and $p$ be the annual payment. These assumptions can be described as follows.

- Each year the balance owed increases in proportion to the balance owed (that proportion being the interest rate).
- Each year the balance owed decreases by the payment made.

We can write a simple equation to model these assumptions as

$$\frac{dB}{dt} = rB - p$$

**Motion due to gravity**. This motion can be described using Newton's second law. This states that the net force acting upon a body equals the mass times the acceleration of the body. The assumptions here are as follows.

- Newton's second law: $\sum F = ma$.
- The weight, or force experienced by an object due to gravity, of mass $m$ is $w = mg$, where $g$ is a constant called the acceleration due to gravity.

If $s(t)$ is the height of an object at time $t$, $m$ is the mass of the object and $g$ is the acceleration due to gravity, then this leads to the simple differential equation

$$m\frac{d^2s}{dt^2} = -mg \quad \Rightarrow \quad \frac{d^2s}{dt^2} = -g$$

Notice that the initial height, $s_0$, and the initial velocity, $v_0$, will have an affect on the position $s(t)$. When there are certain conditions that must be met with some differential equation it is called an **initial value problem**.

$$\begin{cases} \dfrac{d^2s}{dt^2} = -g \\ s(0) = s_0, s'(0) = v_0 \end{cases}$$

Notice that this is an equation which is easily solved by integration.

**Newton's law of heating/cooling**. We can attain a model for the temperature as a function of time of an object placed inside a temperature controlled box (think of a turkey in an oven, etc.) Newton's law of heating/cooling is based on the assumption that

- The rate at which the object heats/cools is proportional to the temperature difference between the object and the medium surrounding it.

This leads to

$$\frac{dT}{dt} = k(T - T_m)$$

where $T_m$ is the (constant) temperature of the medium. Notice that the sign of $k$ determines if the object is heating or cooling.

**Predator-Prey model**. Consider a system consisting of two animal populations where one animal eats the other. We'll denote the population of the prey species as a function of time $t$ by $x(t)$, and the population of the predator species at time $t$ by $y(t)$. We assume the following.

- In the absence of predators, the prey population grows at a rate proportional to its population.
- In the absence of prey, the predator population declines at a rate proportional to its population.
- The rate at which the prey are eaten by predators is proportional to the rate at which the predators and prey interact.
- New predators are born at a rate proportional to the number of prey eaten by the predators.

These assumptions are met by the equations

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned}$$

where $a, b, c, d$ are parameters that represent the birth rate of the prey, the number of prey eaten in predator-prey interactions, the death rate of the predators, and the benefit of eating prey to the predator population. Note that these equations are only one of many possible sets of equations that could fit these assumptions. For instance, changing the first equation to

$$\frac{dx}{dt} = ax - bxy^2$$

also fits the assumptions, and may take into account a huge predator population will have a much more negative effect on the prey than a smaller one would. We will tend to choose the simplest equations that fit the assumptions.

**Simple harmonic oscillator**. Consider a mass on a table attached to a spring. We wish to model the motion of the mass after it is pulled or pushed a bit. Let $y(t)$ be the displacement of the mass from its rest position and we make the following assumptions.

- The only force acting on the mass is that of the spring; that is, we (initially) ignore friction, air resistance, etc.
- The restoring force of the spring is proportional to the displacement, and acts toward the rest position (Hooke's Law).
- The net force on the mass equals the product of the mass and the acceleration (Newton's Second Law).

Thus we have the restoring force is equal to $-ky$, and the sum of the (only) force acting on the mass equals $m\frac{d^2y}{dt^2}$, we have the second order differential equation

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

where $k$ and $m$ are parameters determined by the specific spring and mass.