13.1

1. For $z_1 = -1 + 2i$ and $z_2 = 3 + i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

Solution.

$$z_1 + z_2 = (-1 + 2i) + (3 + i) = (-1 + 3) + (2 + 1)i = \boxed{2 + 3i}$$

$$z_1 - z_2 = (-1 + 2i) - (3 + i) = (-1 - 3) + (2 - 1)i = \boxed{-4 + i}$$

$$z_1 z_2 = (-1 + 2i)(3 + i) = -3 + 2i^2 - 1i + 6i = \boxed{-5 + 5i}$$

$$\frac{z_1}{z_2} = \frac{-1 + 2i}{3 + i} = \frac{(-1 + 2i)(3 - i)}{(3 + i)(3 - i)} = \frac{-1 + 7i}{10} = \boxed{-\frac{1}{10} + \frac{7}{10}i}$$

$$\frac{1}{z_2} = \frac{1}{3 + i} = \frac{3 - i}{(3 + i)(3 - i)} = \frac{3 - i}{10} = \boxed{\frac{3}{10} - \frac{1}{10}i}$$

2. For $z_1 = -2 - 3i$ and $z_2 = -2 + 5i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

Solution.

$$z_1 + z_2 = (-2 - 3i) + (-2 + 5i) = (-2 - 2) + (-3 + 5)i = \boxed{-4 + 2i}$$

$$z_1 - z_2 = (-2 - 3i) - (-2 + 5i) = (-2 + 2) + (-3 - 5)i = \boxed{-8i}$$

$$z_1 z_2 = (-2 - 3i)(-2 + 5i) = 4 - 15i^2 - 10i + 6i = \boxed{19 - 4i}$$

$$\frac{z_1}{z_2} = \frac{-2 - 3i}{-2 + 5i} = \frac{(-2 - 3i)(-2 - 5i)}{(-2 + 5i)(-2 - 5i)} = \frac{-11 + 16i}{29} = \boxed{-\frac{11}{29} + \frac{16}{29}i}$$

$$\frac{1}{z_2} = \frac{1}{-2 + 5i} = \frac{-2 - 5i}{(-2 + 5i)(-2 - 5i)} = \frac{-2 - 5i}{29} = \boxed{-\frac{2}{29} - \frac{5}{29}i}$$

3. For $z_1 = -3 + 3i$ and $z_2 = 3 + 6i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

Solution.

$$z_1 + z_2 = (-3 + 3i) + (3 + 6i) = (-3 + 3) + (3 + 6)i = \boxed{+9i}$$

$$z_1 - z_2 = (-3 + 3i) - (3 + 6i) = (-3 - 3) + (3 - 6)i = \boxed{-6 - 3i}$$

$$z_1 z_2 = (-3 + 3i)(3 + 6i) = -9 + 18i^2 - 18i + 9i = \boxed{-27 - 9i}$$

$$\frac{z_1}{z_2} = \frac{-3 + 3i}{3 + 6i} = \frac{(-3 + 3i)(3 - 6i)}{(3 + 6i)(3 - 6i)} = \frac{9 + 27i}{45} = \boxed{\frac{1}{5} + \frac{3}{5}i}$$

$$\frac{1}{z_2} = \frac{1}{3 + 6i} = \frac{3 - 6i}{(3 + 6i)(3 - 6i)} = \frac{3 - 6i}{45} = \boxed{\frac{1}{15} - \frac{2}{15}i}$$

4. For $z_1 = +4i$ and $z_2 = -3 + 2i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

Solution.

$$z_1 + z_2 = (+4i) + (-3 + 2i) = (0 - 3) + (4 + 2)i = \boxed{-3 + 6i}$$

$$z_1 - z_2 = (+4i) - (-3 + 2i) = (0 + 3) + (4 - 2)i = \boxed{3 + 2i}$$

$$z_1 z_2 = (+4i)(-3 + 2i) = 0 + 8i^2 + 0i - 12i = \boxed{-8 - 12i}$$

$$\frac{z_1}{z_2} = \frac{+4i}{-3 + 2i} = \frac{(+4i)(-3 - 2i)}{(-3 + 2i)(-3 - 2i)} = \frac{8 - 12i}{13} = \boxed{\frac{8}{13} - \frac{12}{13}i}$$

$$\frac{1}{z_2} = \frac{1}{-3 + 2i} = \frac{-3 - 2i}{(-3 + 2i)(-3 - 2i)} = \frac{-3 - 2i}{13} = \boxed{-\frac{3}{13} - \frac{2}{13}i}$$

5. For $z_1 = -4 - i$ and $z_2 = 2 + 3i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

Solution.

$$z_1 + z_2 = (-4 - i) + (2 + 3i) = (-4 + 2) + (-1 + 3)i = \boxed{-2 + 2i}$$

$$z_1 - z_2 = (-4 - i) - (2 + 3i) = (-4 - 2) + (-1 - 3)i = \boxed{-6 - 4i}$$

$$z_1 z_2 = (-4 - i)(2 + 3i) = -8 - 3i^2 - 12i - 2i = \boxed{-5 - 14i}$$

$$\frac{z_1}{z_2} = \frac{-4 - i}{2 + 3i} = \frac{(-4 - i)(2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{-11 + 10i}{13} = \boxed{-\frac{11}{13} + \frac{10}{13}i}$$

$$\frac{1}{z_2} = \frac{1}{2 + 3i} = \frac{2 - 3i}{(2 + 3i)(2 - 3i)} = \frac{2 - 3i}{13} = \boxed{\frac{2}{13} - \frac{3}{13}i}$$

6. For $z_1 = 5 + 2i$ and $z_2 = +i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

Solution.

$$z_1 + z_2 = (5 + 2i) + (+i) = (5 + 0) + (2 + 1)i = \boxed{5 + 3i}$$

$$z_1 - z_2 = (5 + 2i) - (+i) = (5 + 0) + (2 - 1)i = \boxed{5 + i}$$

$$z_1 z_2 = (5 + 2i)(+i) = 0 + 2i^2 + 5i + 0i = \boxed{-2 + 5i}$$

$$\frac{z_1}{z_2} = \frac{5 + 2i}{+i} = \frac{(5 + 2i)(-i)}{(+i)(-i)} = \frac{2 - 5i}{1} = \boxed{2 - 5i}$$

$$\frac{1}{z_2} = \frac{1}{+i} = \frac{-i}{(+i)(-i)} = \frac{-i}{1} = \boxed{0 - 1i}$$

7. For a given complex number $z = x + iy$, find the real and imaginary parts of z^2 and $1/z$.

Solution. To find the real or imaginary part of a quantity, first compute the quantity and work through it so that there is exactly one i in your answer; that is, write it in the form $x + iy$. Then pull out x (for the real part) or y (for the imaginary part).

$$z^2 = (x+iy)^2 = x^2 + 2xyi - y^2 = \underbrace{(x^2 - y^2)}_{Re(z^2)} + i\underbrace{(2xy)}_{Im(z^2)}$$

So $\boxed{Re(z^2) = x^2 - y^2}$, and $\boxed{Im(z^2) = 2xy}$ (note that i is not in the imaginary part).

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

So $\boxed{Re(1/z) = \frac{x}{x^2+y^2}}$ and $\boxed{Im(1/z) = \frac{-y}{x^2+y^2}}$ (don't forget the negative sign).

8. Find the modulus of the complex number $z = 1 + i$.

Solution.

$$|z| = |1+i| = \sqrt{1^2+1^2} = \boxed{\sqrt{2}}.$$

9. Find the modulus of the complex number $z = -2 + 3i$.

Solution.

$$|z| = |-2+3i| = \sqrt{-2^2+3^2} = \boxed{\sqrt{13}}.$$

10. Find the modulus of the complex number $z = -9 + i$.

Solution.

$$|z| = |-9+i| = \sqrt{-9^2+1^2} = \boxed{\sqrt{82}}.$$

11. Find the modulus of the complex number $z = -5 + 4i$.

Solution.

$$|z| = |-5+4i| = \sqrt{-5^2+4^2} = \boxed{\sqrt{41}}.$$

12. Find the modulus of the complex number $z = -5 - 3i$.

Solution.

$$|z| = |-5-3i| = \sqrt{-5^2 + -3^2} = \boxed{\sqrt{34}}.$$

13.2

1. Convert the complex number $z = -2+2i$ to polar form.

   Solution. Note that z is in Quadrant 2. From the ratio

   $$\frac{y}{x} = \frac{2}{-2} = \frac{\sqrt{2}/2}{-\sqrt{2}/2,}$$

   we see that

   $$\sin\theta = \frac{\sqrt{2}}{2}, \quad \cos\theta = -\frac{\sqrt{2}}{2},$$

   and therefore $\theta = \frac{3\pi}{4}$. For the modulus, we compute

   $$|z| = \sqrt{(-2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2},$$

   and so the polar form is

   $$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)}.$$

2. Convert the complex number $z = -4-4i$ to polar form.

   Solution. Note that z is in Quadrant 3. From the ratio

   $$\frac{y}{x} = \frac{-4}{-4} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2,}$$

   we see that

   $$\sin\theta = -\frac{\sqrt{2}}{2}, \quad \cos\theta = -\frac{\sqrt{2}}{2},$$

   and therefore $\theta = \frac{5\pi}{4}$. For the modulus, we compute

   $$|z| = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2},$$

   and so the polar form is

   $$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)}.$$

3. Convert the complex number $z = 2\sqrt{3} - 2i$ to polar form.

Solution. Note that z is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-2}{2\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2,}$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = -\frac{\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).}$$

4. Convert the complex number $z = -1 - i$ to polar form.

Solution. Note that z is in Quadrant 3. From the ratio

$$\frac{y}{x} = \frac{-1}{-1} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2,}$$

we see that

$$\sin\theta = -\frac{\sqrt{2}}{2}, \quad \cos\theta = -\frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{5\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} = \sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right).}$$

5. Convert the complex number $z = 4i$ to polar form.

Solution. Note that z is pure imaginary (real part is zero), and therefore its argument is always $\pm\frac{\pi}{2}$. Since $4 > 0$, $\theta = \frac{\pi}{2}$.
For the modulus, we compute

$$|z| = \sqrt{(0)^2 + (4)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right).}$$

6. Convert the complex number $z = -3i$ to polar form.

Solution. Note that z is pure imaginary (real part is zero), and therefore its argument is always $\pm\frac{\pi}{2}$. Since $-3 < 0$, $\theta = -\frac{\pi}{2}$.
For the modulus, we compute

$$|z| = \sqrt{(0)^2 + (-3)^2} = \sqrt{9} = 3,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right).}$$

7. Convert the complex number $z = -2i$ to polar form.

Solution. Note that z is pure imaginary (real part is zero), and therefore its argument is always $\pm\frac{\pi}{2}$. Since $-2 < 0$, $\theta = -\frac{\pi}{2}$.
For the modulus, we compute

$$|z| = \sqrt{(0)^2 + (-2)^2} = \sqrt{4} = 2,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right).}$$

8. Convert the complex number $z = 2 - 3i$ to polar form.

Solution. Note that z is in Quadrant 4. To find the modulus, we consider the ratio

$$\frac{y}{x} = \frac{-3}{2} = -\frac{3}{2},$$

and so we seek an angle θ such that the numerator is (a multiple of) $\sin\theta$ and the denominator is (a multiple of) $\cos\theta$. There certainly exists such an angle, but it is not among our memorized values on the unit circle. Therefore, the best that we can do is

$$\theta = \arctan\left(-\frac{3}{2}\right).$$

This value is exact since z is in either Quadrant 1 or 4.
For the modulus, we compute

$$|z| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13},$$

and so the polar form is

$$\begin{aligned} z &= |z|(\cos\theta + i\sin\theta) \\ &= \boxed{\sqrt{13}\left(\cos\left(\arctan\left(-\frac{3}{2}\right)\right) + i\sin\left(\arctan\left(-\frac{3}{2}\right)\right)\right).} \end{aligned}$$

9. Convert the complex number $z = 1 - i$ to polar form.

Solution. Note that z is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-1}{1} = \frac{-\sqrt{2}/2}{\sqrt{2}/2,}$$

we see that

$$\sin\theta = -\frac{\sqrt{2}}{2}, \quad \cos\theta = \frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{7\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} = \sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right).}$$

10. Convert the complex number $z = 1 + i$ to polar form.

Solution. Note that z is in Quadrant 1. From the ratio

$$\frac{y}{x} = \frac{1}{1} = \frac{\sqrt{2}/2}{\sqrt{2}/2,}$$

we see that

$$\sin\theta = \frac{\sqrt{2}}{2}, \quad \cos\theta = \frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} = \sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).}$$

11. Convert the complex number $z = 5\sqrt{3} + 5i$ to polar form.

Solution. Note that z is in Quadrant 1. From the ratio

$$\frac{y}{x} = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1/2}{\sqrt{3}/2,}$$

we see that

$$\sin\theta = \frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = \frac{\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(5\sqrt{3})^2 + (5)^2} = \sqrt{100} = 10,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{10\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).}$$

12. Convert the complex number $z = -\sqrt{3} + i$ to polar form.

Solution. Note that z is in Quadrant 2. From the ratio

$$\frac{y}{x} = \frac{1}{-1\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2,}$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = \frac{5\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(-1\sqrt{3})^2 + 1^2} = \sqrt{4} = 2,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right).}$$

13. Convert the complex number $z = -1 + i$ to polar form.

Solution. Note that z is in Quadrant 2. From the ratio

$$\frac{y}{x} = \frac{1}{-1} = \frac{\sqrt{2}/2}{-\sqrt{2}/2,}$$

we see that

$$\sin\theta = \frac{\sqrt{2}}{2}, \quad \cos\theta = -\frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{3\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} = \sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).}$$

14. Convert the complex number $z = 2 + 0i$ to polar form.

Solution. Note that z is actually a real number (imaginary part is zero), and therefore its argument is always either 0 or π. Since $2 > 0$, $\theta = 0$.
For the modulus, we compute

$$|z| = \sqrt{(2)^2 + (0)^2} = \sqrt{4} = 2,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\left(\cos(0) + i\sin(0)\right).}$$

15. Convert the complex number $z = 2 + 3i$ to polar form.

Solution. Note that z is in Quadrant 1. To find the modulus, we consider the ratio

$$\frac{y}{x} = \frac{3}{2} = \frac{3}{2},$$

and so we seek an angle θ such that the numerator is (a multiple of) $\sin\theta$ and the denominator is (a multiple of) $\cos\theta$. There certainly exists such an angle, but it is not among our memorized values on the unit circle. Therefore, the best that we can do is

$$\theta = \arctan\left(\frac{3}{2}\right).$$

This value is exact since z is in either Quadrant 1 or 4.
For the modulus, we compute

$$|z| = \sqrt{(2)^2 + (3)^2} = \sqrt{13},$$

and so the polar form is

$$\begin{aligned} z &= |z|(\cos\theta + i\sin\theta) \\ &= \boxed{\sqrt{13}\left(\cos\left(\arctan\left(\frac{3}{2}\right)\right) + i\sin\left(\arctan\left(\frac{3}{2}\right)\right)\right).} \end{aligned}$$

16. Convert the complex number $z = 4 + 0i$ to polar form.

Solution. Note that z is actually a real number (imaginary part is zero), and therefore its argument is always either 0 or π. Since $4 > 0$, $\theta = 0$.
For the modulus, we compute

$$|z| = \sqrt{(4)^2 + (0)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos(0) + i\sin(0)\right).}$$

17. Convert the complex number $z = 2i$ to polar form.

Solution. Note that z is pure imaginary (real part is zero), and therefore its argument is always $\pm\frac{\pi}{2}$. Since $2 > 0$, $\theta = \frac{\pi}{2}$.
For the modulus, we compute

$$|z| = \sqrt{(0)^2 + (2)^2} = \sqrt{4} = 2,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right).}$$

18. Convert the complex number $z = -5\sqrt{3} + 5i$ to polar form.

Solution. Note that z is in Quadrant 2. From the ratio

$$\frac{y}{x} = \frac{5}{-5\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2},$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = \frac{5\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(-5\sqrt{3})^2 + 5^2} = \sqrt{100} = 10,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{10\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right).}$$

19. Convert the complex number $z = \sqrt{3} - i$ to polar form.

Solution. Note that z is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-1}{1\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2},$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = -\frac{\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(1\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).}$$

20. Convert the complex number $z = -2 + 0i$ to polar form.

Solution. Note that z is actually a real number (imaginary part is zero), and therefore its argument is always either 0 or π. Since $-2 < 0$, $\theta = \pi$.
For the modulus, we compute

$$|z| = \sqrt{(-2)^2 + (0)^2} = \sqrt{4} = 2,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\left(\cos(\pi) + i\sin(\pi)\right).}$$

21. Convert the complex number $z = -2\sqrt{3} - 2i$ to polar form.

Solution. Note that z is in Quadrant 3. From the ratio

$$\frac{y}{x} = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{-1/2}{-\sqrt{3}/2},$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = -\frac{\sqrt{3}}{2},$$

and therefore $\theta = -\frac{5\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)}.$$

22. Convert the complex number $z = -4 + 4i$ to polar form.

Solution. Note that z is in Quadrant 2. From the ratio

$$\frac{y}{x} = \frac{4}{-4} = \frac{\sqrt{2}/2}{-\sqrt{2}/2},$$

we see that

$$\sin\theta = \frac{\sqrt{2}}{2}, \quad \cos\theta = -\frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{3\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(-4)^2 + (4)^2} = \sqrt{32} = 4\sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)}.$$

23. Convert the complex number $z = 3 + 3i$ to polar form.

Solution. Note that z is in Quadrant 1. From the ratio

$$\frac{y}{x} = \frac{3}{3} = \frac{\sqrt{2}/2}{\sqrt{2}/2},$$

we see that

$$\sin\theta = \frac{\sqrt{2}}{2}, \quad \cos\theta = \frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{3\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)}.$$

24. Convert the complex number $z = -2 - 2\sqrt{3}i$ to polar form.

Solution. Note that z is in Quadrant 3. From the ratio

$$\frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \frac{-\sqrt{3}}{-1} = \frac{-\sqrt{3}/2}{-1/2},$$

we see that

$$\sin\theta = -\frac{\sqrt{3}}{2}, \quad \cos\theta = -\frac{1}{2},$$

and therefore $\theta = \frac{4\pi}{3}$. For the modulus, we compute

$$|z| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right).}$$

25. Convert the complex number $z = -3\sqrt{3} - 3i$ to polar form.

Solution. Note that z is in Quadrant 3. From the ratio

$$\frac{y}{x} = \frac{-3}{-3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{-1/2}{-\sqrt{3}/2,}$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = -\frac{\sqrt{3}}{2},$$

and therefore $\theta = -\frac{5\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(-3\sqrt{3})^2 + (-3)^2} = \sqrt{36} = 6,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{6\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right).}$$

26. Convert the complex number $z = 3\sqrt{3} - 3i$ to polar form.

Solution. Note that z is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-3}{3\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2,}$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = -\frac{\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(3\sqrt{3})^2 + (-3)^2} = \sqrt{36} = 6,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{6\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).}$$

27. Convert the complex number $z = 2 - 2\sqrt{3}i$ to polar form.

Solution. Note that z is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-2\sqrt{3}}{2} = \frac{-\sqrt{3}}{1} = \frac{-\sqrt{3}/2}{1/2},$$

we see that

$$\sin\theta = -\frac{\sqrt{3}}{2}, \quad \cos\theta = \frac{1}{2},$$

and therefore $\theta = -\frac{\pi}{3}$. For the modulus, we compute

$$|z| = \sqrt{(-2\sqrt{3})^2 + (2)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right).}$$

28. Convert the complex number $z = 2 + 2\sqrt{3}i$ to polar form.

Solution. Note that z is in Quadrant 1. From the ratio

$$\frac{y}{x} = \frac{2\sqrt{3}}{2} = \frac{\sqrt{3}}{1} = \frac{\sqrt{3}/2}{1/2},$$

we see that

$$\sin\theta = \frac{\sqrt{3}}{2}, \quad \cos\theta = \frac{1}{2},$$

and therefore $\theta = \frac{\pi}{3}$. For the modulus, we compute

$$|z| = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).}$$

29. Convert the complex number $z = -2 - 3i$ to polar form.

Solution. Note that z is in Quadrant 3. To find the modulus, we consider the ratio

$$\frac{y}{x} = \frac{-3}{-2} = \frac{3}{2},$$

and so we seek an angle θ such that the numerator is (a multiple of) $\sin\theta$ and the denominator is (a multiple of) $\cos\theta$. There certainly exists such an angle, but it is not among our memorized values on the unit circle. Therefore, we consider $\arctan\left(\frac{3}{2}\right)$. Since the range of the arctangent function is only in quadrants 1 and 4, we need to add π to this value to get the true argument of $z = -2 - 3i$, so we have

$$\theta = \arctan\left(\frac{3}{2}\right) + \pi.$$

Remember this goes outside of the arctangent function. For the modulus, we compute

$$|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13},$$

and so the polar form is

$$z \quad = \quad \boxed{\sqrt{13}\left(\cos\left(\arctan\left(\frac{3}{2}\right) + \pi\right) + i\sin\left(\arctan\left(\frac{3}{2}\right) + \pi\right)\right).}$$

30. Convert the complex number $z = -4\sqrt{3} + 4i$ to polar form.

Solution. Note that z is in Quadrant 2. From the ratio

$$\frac{y}{x} = \frac{4}{-4\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2,}$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = \dfrac{5\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(-4\sqrt{3})^2 + 4^2} = \sqrt{64} = 8,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{8\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right).}$$

31. Convert the complex number $z = 4\sqrt{3} - 4i$ to polar form.

Solution. Note that z is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-4}{4\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2,}$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = -\frac{\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{64} = 8,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{8\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)}.$$

32. Convert the complex number $z = \sqrt{3} + i$ to polar form.

Solution. Note that z is in Quadrant 1. From the ratio

$$\frac{y}{x} = \frac{1}{1\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1/2}{\sqrt{3}/2,}$$

we see that

$$\sin\theta = \frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = \frac{\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(1\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)}.$$

33. Convert the complex number $z = 5 + 5i$ to polar form.

Solution. Note that z is in Quadrant 1. From the ratio

$$\frac{y}{x} = \frac{5}{5} = \frac{\sqrt{2}/2}{\sqrt{2}/2,}$$

we see that

$$\sin\theta = \frac{\sqrt{2}}{2}, \quad \cos\theta = \frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(5)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{5\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)}.$$

34. Convert the complex number $z = -2 + 2\sqrt{3}i$ to polar form.

Solution. Note that z is in Quadrant 2. From the ratio

$$\frac{y}{x} = \frac{2\sqrt{3}}{-2} = \frac{\sqrt{3}}{-1} = \frac{\sqrt{3}/2}{-1/2},$$

we see that

$$\sin\theta = \frac{\sqrt{3}}{2}, \quad \cos\theta = -\frac{1}{2},$$

and therefore $\theta = \frac{2\pi}{3}$. For the modulus, we compute

$$|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)}.$$

35. Convert the complex number $z = -\sqrt{3} - i$ to polar form.

Solution. Note that z is in Quadrant 3. From the ratio

$$\frac{y}{x} = \frac{-1}{-1\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{-1/2}{-\sqrt{3}/2,}$$

we see that

$$\sin\theta = -\frac{1}{2}, \quad \cos\theta = -\frac{\sqrt{3}}{2},$$

and therefore $\theta = -\frac{5\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(-1\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)}.$$

36. Convert the complex number $z = -2 + 3i$ to polar form.

Solution. Note that z is in Quadrant 2. To find the modulus, we consider the ratio

$$\frac{y}{x} = \frac{3}{-2} = -\frac{3}{2},$$

and so we seek an angle θ such that the numerator is (a multiple of) $\sin\theta$ and the denominator is (a multiple of) $\cos\theta$. There certainly exists such an angle, but it is not among our memorized values on the unit circle. Therefore, we consider $\arctan\left(-\frac{3}{2}\right)$. Since the range of the arctangent function is only in quadrants 1 and 4, we need to add π to this value to get the true argument of $z = -2 + 3i$, so we have

$$\theta = \arctan\left(-\frac{3}{2}\right) + \pi.$$

Remember this goes outside of the arctangent function. For the modulus, we compute

$$|z| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13},$$

and so the polar form is

$$z = \boxed{\sqrt{13}\left(\cos\left(\arctan\left(-\frac{3}{2}\right) + \pi\right) + i\sin\left(\arctan\left(-\frac{3}{2}\right) + \pi\right)\right).}$$

37. Convert the complex number $z = -3 + 0i$ to polar form.

Solution. Note that z is actually a real number (imaginary part is zero), and therefore its argument is always either 0 or π. Since $-3 < 0$, $\theta = \pi$.
For the modulus, we compute

$$|z| = \sqrt{(-3)^2 + (0)^2} = \sqrt{9} = 3,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{3\left(\cos(\pi) + i\sin(\pi)\right).}$$

38. Convert the complex number $z = 2\sqrt{3} + 2i$ to polar form.

Solution. Note that z is in Quadrant 1. From the ratio

$$\frac{y}{x} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1/2}{\sqrt{3}/2,}$$

we see that

$$\sin\theta = \frac{1}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2},$$

and therefore $\theta = \frac{\pi}{6}$. For the modulus, we compute

$$|z| = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{16} = 4,$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).}$$

39. Convert the complex number $z = 2 - 2i$ to polar form.

Solution. Note that z is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-2}{2} = \frac{-\sqrt{2}/2}{\sqrt{2}/2,}$$

we see that

$$\sin\theta = -\frac{\sqrt{2}}{2}, \quad \cos\theta = \frac{\sqrt{2}}{2},$$

and therefore $\theta = \dfrac{7\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{2\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right).}$$

40. Convert the complex number $z = 5 - 5i$ to polar form.

Solution. Note that z is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-5}{5} = \frac{-\sqrt{2}/2}{\sqrt{2}/2,}$$

we see that

$$\sin\theta = -\frac{\sqrt{2}}{2}, \quad \cos\theta = \frac{\sqrt{2}}{2},$$

and therefore $\theta = \dfrac{7\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(5)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2},$$

and so the polar form is

$$z = |z|(\cos\theta + i\sin\theta) = \boxed{5\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right).}$$

41. For the complex numbers $z_1 = 2\left(\cos\left(\frac{1}{7}\pi\right) + i\sin\left(\frac{1}{7}\pi\right)\right)$ and $z_2 = 3\left(\cos\left(\frac{2}{5}\pi\right) + i\sin\left(\frac{2}{5}\pi\right)\right)$, compute $z_1 z_2$, $\dfrac{z_1}{z_2}$, z_1^2, and z_2^3.

Solution.

$$z_1 z_2 = (2)(3)\,\text{cis}\left(\frac{1}{7}\pi + \frac{2}{5}\pi\right) = 6\text{cis}\left(\frac{19}{35}\pi\right)$$

$$\frac{z_1}{z_2} = \frac{2}{3}\text{cis}\left(\frac{1}{7}\pi - \frac{2}{5}\pi\right) = \frac{2}{3}\text{cis}\left(-\frac{9}{35}\pi\right)$$

$$z_1^2 = (2^2)\,\text{cis}\left((2)\,\frac{1}{7}\pi\right) = 4\text{cis}\left(\frac{2}{7}\pi\right)$$

$$z_2^3 = (3^3)\,\text{cis}\left((3)\,\frac{2}{5}\pi\right) = 27\text{cis}\left(\frac{6}{5}\pi\right)$$

42. For the complex numbers $z_1 = 2\left(\cos\left(\frac{-2}{7}\pi\right) + i\sin\left(\frac{-2}{7}\pi\right)\right)$ and $z_2 = 2\left(\cos\left(\frac{2}{11}\pi\right) + i\sin\left(\frac{2}{11}\pi\right)\right)$, compute z_1z_2, $\frac{z_1}{z_2}$, z_1^5, and z_2^4.

Solution.

$$z_1z_2 = (2)(2)\,\text{cis}\left(\frac{-2}{7}\pi + \frac{2}{11}\pi\right) = 4\text{cis}\left(-\frac{8}{77}\pi\right)$$

$$\frac{z_1}{z_2} = \frac{2}{2}\text{cis}\left(\frac{-2}{7}\pi - \frac{2}{11}\pi\right) = 1\text{cis}\left(-\frac{36}{77}\pi\right)$$

$$z_1^5 = \left(2^5\right)\text{cis}\left((5)\,\frac{-2}{7}\pi\right) = 32\text{cis}\left(-\frac{10}{7}\pi\right)$$

$$z_2^4 = \left(2^4\right)\text{cis}\left((4)\,\frac{2}{11}\pi\right) = 16\text{cis}\left(\frac{8}{11}\pi\right)$$

43. For the complex numbers $z_1 = 3\left(\cos\left(\frac{-1}{11}\pi\right) + i\sin\left(\frac{-1}{11}\pi\right)\right)$ and $z_2 = 2\left(\cos\left(\frac{5}{13}\pi\right) + i\sin\left(\frac{5}{13}\pi\right)\right)$, compute z_1z_2, $\frac{z_1}{z_2}$, z_1^2, and z_2^4.

Solution.

$$z_1z_2 = (3)(2)\,\text{cis}\left(\frac{-1}{11}\pi + \frac{5}{13}\pi\right) = 6\text{cis}\left(\frac{42}{143}\pi\right)$$

$$\frac{z_1}{z_2} = \frac{3}{2}\text{cis}\left(\frac{-1}{11}\pi - \frac{5}{13}\pi\right) = \frac{3}{2}\text{cis}\left(-\frac{68}{143}\pi\right)$$

$$z_1^2 = \left(3^2\right)\text{cis}\left((2)\,\frac{-1}{11}\pi\right) = 9\text{cis}\left(-\frac{2}{11}\pi\right)$$

$$z_2^4 = \left(2^4\right)\text{cis}\left((4)\,\frac{5}{13}\pi\right) = 16\text{cis}\left(\frac{20}{13}\pi\right)$$

44. For the complex numbers $z_1 = 2\left(\cos\left(\frac{4}{5}\pi\right) + i\sin\left(\frac{4}{5}\pi\right)\right)$ and $z_2 = \left(\cos\left(\frac{3}{7}\pi\right) + i\sin\left(\frac{3}{7}\pi\right)\right)$, compute z_1z_2, $\frac{z_1}{z_2}$, z_1^4, and z_2^3.

Solution.

$$z_1z_2 = (2)(1)\,\text{cis}\left(\frac{4}{5}\pi + \frac{3}{7}\pi\right) = 2\text{cis}\left(\frac{43}{35}\pi\right)$$

$$\frac{z_1}{z_2} = \frac{2}{1}\text{cis}\left(\frac{4}{5}\pi - \frac{3}{7}\pi\right) = 2\text{cis}\left(\frac{13}{35}\pi\right)$$

$$z_1^4 = \left(2^4\right)\text{cis}\left((4)\,\frac{4}{5}\pi\right) = 16\text{cis}\left(\frac{16}{5}\pi\right)$$

$$z_2^3 = \left(1^3\right)\text{cis}\left((3)\,\frac{3}{7}\pi\right) = \text{cis}\left(\frac{9}{7}\pi\right)$$

45. For the complex numbers $z_1 = \left(\cos\left(\frac{-3}{11}\pi\right) + i\sin\left(\frac{-3}{11}\pi\right)\right)$ and $z_2 = \left(\cos\left(\frac{-3}{7}\pi\right) + i\sin\left(\frac{-3}{7}\pi\right)\right)$, compute z_1z_2, $\frac{z_1}{z_2}$, z_1^2, and z_2^4.

Solution.

$$z_1z_2 = (1)(1)\,\mathrm{cis}\left(\frac{-3}{11}\pi + \frac{-3}{7}\pi\right) = \mathrm{cis}\left(-\frac{54}{77}\pi\right)$$

$$\frac{z_1}{z_2} = \frac{1}{1}\mathrm{cis}\left(\frac{-3}{11}\pi - \frac{-3}{7}\pi\right) = 1\mathrm{cis}\left(\frac{12}{77}\pi\right)$$

$$z_1^2 = (1^2)\,\mathrm{cis}\left((2)\,\frac{-3}{11}\pi\right) = \mathrm{cis}\left(-\frac{6}{11}\pi\right)$$

$$z_2^4 = (1^4)\,\mathrm{cis}\left((4)\,\frac{-3}{7}\pi\right) = \mathrm{cis}\left(-\frac{12}{7}\pi\right)$$

46. Find all cube roots of $z = 2\left(\cos\left(\frac{3}{10}\pi\right) + i\sin\left(\frac{3}{10}\pi\right)\right)$. You may leave your answer in polar form.

Solution. The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z} = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case, since z is given in polar form,

$$|\sqrt[3]{z} = \sqrt[3]{2}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{10}\pi + 2k\pi}{3}, \quad k = 0, 1, 2.$$

Then for each value of k,

$k = 0$: $\quad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{10}\pi}{3} = \frac{1}{10}\pi \qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\mathrm{cis}\left(\frac{1}{10}\pi\right)\right)}$

$k = 1$: $\quad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{10}\pi + 2\pi}{3} = \frac{23}{30}\pi \qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\mathrm{cis}\left(\frac{23}{30}\pi\right)\right)}$

$k = 2$: $\quad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{10}\pi + 4\pi}{3} = \frac{43}{30}\pi \qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\mathrm{cis}\left(\frac{43}{30}\pi\right)\right)}$

47. Find all cube roots of $z = 2\left(\cos\left(\frac{3}{11}\pi\right) + i\sin\left(\frac{3}{11}\pi\right)\right)$. You may leave your answer in polar form.

Solution. The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z} = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case, since z is given in polar form,

$$|\sqrt[3]{z} = \sqrt[3]{2}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{11}\pi + 2k\pi}{3}, \quad k = 0, 1, 2.$$

Then for each value of k,

$k = 0$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{11}\pi}{3} = \frac{1}{11}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\operatorname{cis}\left(\frac{1}{11}\pi\right)\right)}$

$k = 1$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{11}\pi + 2\pi}{3} = \frac{25}{33}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\operatorname{cis}\left(\frac{25}{33}\pi\right)\right)}$

$k = 2$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{11}\pi + 4\pi}{3} = \frac{47}{33}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\operatorname{cis}\left(\frac{47}{33}\pi\right)\right)}$

48. Find all cube roots of $z = 2\left(\cos\left(\frac{2}{7}\pi\right) + i\sin\left(\frac{2}{7}\pi\right)\right)$. You may leave your answer in polar form.

Solution. The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z} = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case, since z is given in polar form,

$$|\sqrt[3]{z} = \sqrt[3]{2}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{2}{7}\pi + 2k\pi}{3}, \quad k = 0, 1, 2.$$

Then for each value of k,

$k = 0$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{2}{7}\pi}{3} = \frac{2}{21}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\operatorname{cis}\left(\frac{2}{21}\pi\right)\right)}$

$k = 1$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{2}{7}\pi + 2\pi}{3} = \frac{16}{21}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\operatorname{cis}\left(\frac{16}{21}\pi\right)\right)}$

$k = 2$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{2}{7}\pi + 4\pi}{3} = \frac{10}{7}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\operatorname{cis}\left(\frac{10}{7}\pi\right)\right)}$

49. Find all cube roots of $z = 3\left(\cos\left(\frac{3}{5}\pi\right) + i\sin\left(\frac{3}{5}\pi\right)\right)$. You may leave your answer in polar form.

Solution. The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z} = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case, since z is given in polar form,

$$|\sqrt[3]{z} = \sqrt[3]{3}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{5}\pi + 2k\pi}{3}, \quad k = 0, 1, 2.$$

Then for each value of k,

$k = 0$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{5}\pi}{3} = \frac{1}{5}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{3}\left(\text{cis}\left(\frac{1}{5}\pi\right)\right)}$

$k = 1$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{5}\pi + 2\pi}{3} = \frac{13}{15}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{3}\left(\text{cis}\left(\frac{13}{15}\pi\right)\right)}$

$k = 2$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{5}\pi + 4\pi}{3} = \frac{23}{15}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{3}\left(\text{cis}\left(\frac{23}{15}\pi\right)\right)}$

50. Find all cube roots of $z = 2\left(\cos\left(\frac{3}{7}\pi\right) + i\sin\left(\frac{3}{7}\pi\right)\right)$. You may leave your answer in polar form.

Solution. The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z} = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case, since z is given in polar form,

$$|\sqrt[3]{z} = \sqrt[3]{2}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{7}\pi + 2k\pi}{3}, \quad k = 0, 1, 2.$$

Then for each value of k,

$k = 0$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{7}\pi}{3} = \frac{1}{7}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{1}{7}\pi\right)\right)}$

$k = 1$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{7}\pi + 2\pi}{3} = \frac{17}{21}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{17}{21}\pi\right)\right)}$

$k = 2$: $\arg\left(\sqrt[3]{z}\right) = \frac{\frac{3}{7}\pi + 4\pi}{3} = \frac{31}{21}\pi$ $\qquad \sqrt[3]{z} = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{31}{21}\pi\right)\right)}$

51. Find all square roots of $z = -1 - 4i$.

Solution. We first convert to polar coordindates. In Quadrant 3, the polar form of z is

$$z = \sqrt{17}\,(\text{cis}\,(\arctan(4) + \pi))$$

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of $\pm\pi$ to get the argument for z. The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{17}} = \sqrt[4]{17}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(4) + \pi + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{17}\left(\text{cis}\left(\frac{\arctan(4) + \pi}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{17}\left(\text{cis}\left(\frac{\arctan(4) + 3\pi}{2}\right)\right)}$$

52. Find all square roots of $z = 3 - i$.

Solution. We first convert to polar coordindates. In Quadrant 4, the polar form of z is

$$z = \sqrt{10}\,(\cos\,(\arctan(-1/3)) + i\sin\,(\arctan(-1/3)))$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{10}} = \sqrt[4]{10}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(-1/3) + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{10}\left(\text{cis}\left(\frac{\arctan(-1/3)}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{10}\left(\text{cis}\left(\frac{\arctan(-1/3) + 2\pi}{2}\right)\right)}$$

53. Find all square roots of $z = 1 + 5i$.

Solution. We first convert to polar coordindates. In Quadrant 1, the polar form of z is

$$z = \sqrt{26}\left(\cos\left(\arctan(5)\right) + i\sin\left(\arctan(5)\right)\right)$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{26}} = \sqrt[4]{26}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(5) + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{26}\left(\text{cis}\left(\frac{\arctan(5)}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{26}\left(\text{cis}\left(\frac{\arctan(5) + 2\pi}{2}\right)\right)}$$

54. Find all square roots of $z = 4 + i$.

Solution. We first convert to polar coordindates. In Quadrant 1, the polar form of z is

$$z = \sqrt{17}\left(\cos\left(\arctan(1/4)\right) + i\sin\left(\arctan(1/4)\right)\right)$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{17}} = \sqrt[4]{17}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(1/4) + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{17}\left(\text{cis}\left(\frac{\arctan(1/4)}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{17}\left(\text{cis}\left(\frac{\arctan(1/4) + 2\pi}{2}\right)\right)}$$

55. Find all square roots of $z = -5 - i$.

Solution. We first convert to polar coordindates. In Quadrant 3, the polar form of z is

$$z = \sqrt{26}\,(\operatorname{cis}(\arctan(1/5) + \pi))$$

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of $\pm\pi$ to get the argument for z. The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{26}} = \sqrt[4]{26}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(1/5) + \pi + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{26}\left(\operatorname{cis}\left(\frac{\arctan(1/5) + \pi}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{26}\left(\operatorname{cis}\left(\frac{\arctan(1/5) + 3\pi}{2}\right)\right)}$$

56. Find all square roots of $z = 2 - 5i$.

Solution. We first convert to polar coordindates. In Quadrant 4, the polar form of z is

$$z = \sqrt{29}\,(\cos(\arctan(-5/2)) + i\sin(\arctan(-5/2)))$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{29}} = \sqrt[4]{29}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(-5/2) + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{29}\left(\operatorname{cis}\left(\frac{\arctan(-5/2)}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{29}\left(\operatorname{cis}\left(\frac{\arctan(-5/2) + 2\pi}{2}\right)\right)}$$

57. Find all square roots of $z = -2 + 3i$.

Solution. We first convert to polar coordindates. In Quadrant 2, the polar form of z is

$$z = \sqrt{13}\,(\text{cis}\,(\arctan(-3/2) + \pi))$$

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of $\pm\pi$ to get the argument for z. The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{13}} = \sqrt[4]{13}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(-3/2) + \pi + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{13}\left(\text{cis}\left(\frac{\arctan(-3/2) + \pi}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{13}\left(\text{cis}\left(\frac{\arctan(-3/2) + 3\pi}{2}\right)\right)}$$

58. Find all square roots of $z = -3 + i$.

Solution. We first convert to polar coordindates. In Quadrant 2, the polar form of z is

$$z = \sqrt{10}\,(\text{cis}\,(\arctan(-1/3) + \pi))$$

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of $\pm\pi$ to get the argument for z. The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{10}} = \sqrt[4]{10}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(-1/3) + \pi + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{10}\left(\text{cis}\left(\frac{\arctan(-1/3) + \pi}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{10}\left(\text{cis}\left(\frac{\arctan(-1/3) + 3\pi}{2}\right)\right)}$$

59. Find all square roots of $z = 2 + 7i$.

Solution. We first convert to polar coordindates. In Quadrant 1, the polar form of z is

$$z = \sqrt{53}\left(\cos\left(\arctan(7/2)\right) + i\sin\left(\arctan(7/2)\right)\right)$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{53}} = \sqrt[4]{53}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(7/2) + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{53}\left(\text{cis}\left(\frac{\arctan(7/2)}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{53}\left(\text{cis}\left(\frac{\arctan(7/2) + 2\pi}{2}\right)\right)}$$

60. Find all square roots of $z = 2 + 3i$.

Solution. We first convert to polar coordindates. In Quadrant 1, the polar form of z is

$$z = \sqrt{13}\left(\cos\left(\arctan(3/2)\right) + i\sin\left(\arctan(3/2)\right)\right)$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{13}} = \sqrt[4]{13}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(3/2) + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{13}\left(\text{cis}\left(\frac{\arctan(3/2)}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{13}\left(\text{cis}\left(\frac{\arctan(3/2) + 2\pi}{2}\right)\right)}$$

61. Find all square roots of $z = -1 + 5i$.

Solution. We first convert to polar coordindates. In Quadrant 2, the polar form of z is

$$z = \sqrt{26}\,(\text{cis}\,(\arctan(-5) + \pi))$$

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of $\pm\pi$ to get the argument for z. The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{26}} = \sqrt[4]{26}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(-5) + \pi + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{26}\left(\text{cis}\left(\frac{\arctan(-5) + \pi}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{26}\left(\text{cis}\left(\frac{\arctan(-5) + 3\pi}{2}\right)\right)}$$

62. Find all square roots of $z = 2 - 7i$.

Solution. We first convert to polar coordindates. In Quadrant 4, the polar form of z is

$$z = \sqrt{53}\,(\cos\,(\arctan(-7/2)) + i\sin\,(\arctan(-7/2)))$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{53}} = \sqrt[4]{53}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(-7/2) + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{53}\left(\text{cis}\left(\frac{\arctan(-7/2)}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{53}\left(\text{cis}\left(\frac{\arctan(-7/2) + 2\pi}{2}\right)\right)}$$

63. Find all square roots of $z = 1 - 6i$.

Solution. We first convert to polar coordindates. In Quadrant 4, the polar form of z is

$$z = \sqrt{37}\left(\cos\left(\arctan(-6)\right) + i\sin\left(\arctan(-6)\right)\right)$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{37}} = \sqrt[4]{37}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(-6) + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{37}\left(\operatorname{cis}\left(\frac{\arctan(-6)}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{37}\left(\operatorname{cis}\left(\frac{\arctan(-6) + 2\pi}{2}\right)\right)}$$

64. Find all square roots of $z = -2 + 7i$.

Solution. We first convert to polar coordindates. In Quadrant 2, the polar form of z is

$$z = \sqrt{53}\left(\operatorname{cis}\left(\arctan(-7/2) + \pi\right)\right)$$

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of $\pm\pi$ to get the argument for z. The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{53}} = \sqrt[4]{53}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(-7/2) + \pi + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{53}\left(\operatorname{cis}\left(\frac{\arctan(-7/2) + \pi}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{53}\left(\operatorname{cis}\left(\frac{\arctan(-7/2) + 3\pi}{2}\right)\right)}$$

65. Find all square roots of $z = -2 - 7i$.

Solution. We first convert to polar coordindates. In Quadrant 3, the polar form of z is

$$z = \sqrt{53}\,(\text{cis}\,(\arctan(7/2) + \pi))$$

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of $\pm\pi$ to get the argument for z. The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{\sqrt{53}} = \sqrt[4]{53}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arctan(7/2) + \pi + 2k\pi}{2}, \quad k = 0, 1.$$

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{53}\left(\text{cis}\left(\frac{\arctan(7/2) + \pi}{2}\right)\right)}$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \boxed{\sqrt[4]{53}\left(\text{cis}\left(\frac{\arctan(7/2) + 3\pi}{2}\right)\right)}$$

66. Find all complex square roots of $z = 1 + \sqrt{3}i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{2}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{3} + 2k\pi}{2}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt{z} = \sqrt{2}\left(\text{cis}\left(\frac{\frac{\pi}{3} + 0}{2}\right)\right) = \boxed{\sqrt{2}\left(\text{cis}\left(\frac{\pi}{6}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt{z} = \sqrt{2}\left(\text{cis}\left(\frac{\frac{\pi}{3} + \frac{6\pi}{3}}{2}\right)\right) = \boxed{\sqrt{2}\left(\text{cis}\left(\frac{7\pi}{6}\right)\right)}$$

67. Find all complex square roots of $z = 4 + 4i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 4\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{4\sqrt{2}} = \sqrt{4}\sqrt[4]{2}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{4} + 2k\pi}{2}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt{z} = \sqrt{4}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\frac{\pi}{4} + 0}{2}\right)\right) = \boxed{\sqrt{4}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\pi}{8}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt{z} = \sqrt{4}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\frac{\pi}{4} + \frac{8\pi}{4}}{2}\right)\right) = \boxed{\sqrt{4}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{9\pi}{8}\right)\right)}$$

68. Find all complex square roots of $z = 2 + 2i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{2\sqrt{2}} = \sqrt{2}\sqrt[4]{2}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{4} + 2k\pi}{2}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt{z} = \sqrt{2}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\frac{\pi}{4} + 0}{2}\right)\right) = \boxed{\sqrt{2}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\pi}{8}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt{z} = \sqrt{2}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\frac{\pi}{4} + \frac{8\pi}{4}}{2}\right)\right) = \boxed{\sqrt{2}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{9\pi}{8}\right)\right)}$$

69. Find all complex cube roots of $z = 3 + 3\sqrt{3}i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z}| = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case,

$$|\sqrt[3]{z}| = \sqrt[3]{6}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{\pi}{3} + 2k\pi}{3}, \quad k = 0, 1, 2.$$

$$k = 0: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{6}\left(\text{cis}\left(\frac{\frac{\pi}{3} + 0}{3}\right)\right) = \boxed{\sqrt[3]{6}\left(\text{cis}\left(\frac{\pi}{9}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{6}\left(\text{cis}\left(\frac{\frac{\pi}{3} + \frac{6\pi}{3}}{3}\right)\right) = \boxed{\sqrt[3]{6}\left(\text{cis}\left(\frac{7\pi}{9}\right)\right)}$$

$$k = 2: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{6}\left(\text{cis}\left(\frac{\frac{\pi}{3} + \frac{12\pi}{3}}{3}\right)\right) = \boxed{\sqrt[3]{6}\left(\text{cis}\left(\frac{13\pi}{9}\right)\right)}$$

70. Find all complex cube roots of $z = 1 + \sqrt{3}i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z}| = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case,

$$|\sqrt[3]{z}| = \sqrt[3]{2}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{\pi}{3} + 2k\pi}{3}, \quad k = 0, 1, 2.$$

$$k = 0: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{2}\left(\text{cis}\left(\frac{\frac{\pi}{3} + 0}{3}\right)\right) = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{\pi}{9}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{2}\left(\text{cis}\left(\frac{\frac{\pi}{3} + \frac{6\pi}{3}}{3}\right)\right) = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{7\pi}{9}\right)\right)}$$

$$k = 2: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{2}\left(\text{cis}\left(\frac{\frac{\pi}{3} + \frac{12\pi}{3}}{3}\right)\right) = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{13\pi}{9}\right)\right)}$$

71. Find all complex square roots of $z = 3\sqrt{3} + 3i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 6\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{6}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{6} + 2k\pi}{2}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt{z} = \sqrt{6}\left(\text{cis}\left(\frac{\frac{\pi}{6} + 0}{2}\right)\right) = \boxed{\sqrt{6}\left(\text{cis}\left(\frac{\pi}{12}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt{z} = \sqrt{6}\left(\text{cis}\left(\frac{\frac{\pi}{6} + \frac{12\pi}{6}}{2}\right)\right) = \boxed{\sqrt{6}\left(\text{cis}\left(\frac{13\pi}{12}\right)\right)}$$

72. Find all complex cube roots of $z = \sqrt{3} + i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).$$

The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z}| = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case,

$$|\sqrt[3]{z}| = \sqrt[3]{2}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{\pi}{6} + 2k\pi}{3}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{2}\left(\text{cis}\left(\frac{\frac{\pi}{6} + 0}{3}\right)\right) = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{\pi}{18}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{2}\left(\text{cis}\left(\frac{\frac{\pi}{6} + \frac{12\pi}{6}}{3}\right)\right) = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{13\pi}{18}\right)\right)}$$

$$k = 2: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{2}\left(\text{cis}\left(\frac{\frac{\pi}{6} + \frac{24\pi}{6}}{3}\right)\right) = \boxed{\sqrt[3]{2}\left(\text{cis}\left(\frac{25\pi}{18}\right)\right)}$$

73. Find all complex cube roots of $z = 3 + 3i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 3\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z}| = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case,

$$|\sqrt[3]{z}| = \sqrt[3]{3\sqrt{2}} = \sqrt[3]{3}\sqrt[6]{2}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{4} + 2k\pi}{3}, \quad k = 0, 1, 2.$$

$$k = 0: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{3}\sqrt[6]{2}\left(\text{cis}\left(\frac{\frac{\pi}{4} + 0}{3}\right)\right) = \boxed{\sqrt[3]{3}\sqrt[6]{2}\left(\text{cis}\left(\frac{\pi}{12}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{3}\sqrt[6]{2}\left(\text{cis}\left(\frac{\frac{\pi}{4} + \frac{8\pi}{4}}{3}\right)\right) = \boxed{\sqrt[3]{3}\sqrt[6]{2}\left(\text{cis}\left(\frac{9\pi}{12}\right)\right)}$$

$$k = 2: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{3}\sqrt[6]{2}\left(\text{cis}\left(\frac{\frac{\pi}{4} + \frac{16\pi}{4}}{3}\right)\right) = \boxed{\sqrt[3]{3}\sqrt[6]{2}\left(\text{cis}\left(\frac{17\pi}{12}\right)\right)}$$

74. Find all complex cube roots of $z = 3\sqrt{3} + 3i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 6\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).$$

The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z}| = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case,

$$|\sqrt[3]{z}| = \sqrt[3]{6}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\frac{\pi}{6} + 2k\pi}{3}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{6}\left(\text{cis}\left(\frac{\frac{\pi}{6} + 0}{3}\right)\right) = \boxed{\sqrt[3]{6}\left(\text{cis}\left(\frac{\pi}{18}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{6}\left(\text{cis}\left(\frac{\frac{\pi}{6} + \frac{12\pi}{6}}{3}\right)\right) = \boxed{\sqrt[3]{6}\left(\text{cis}\left(\frac{13\pi}{18}\right)\right)}$$

$$k = 2: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{6}\left(\text{cis}\left(\frac{\frac{\pi}{6} + \frac{24\pi}{6}}{3}\right)\right) = \boxed{\sqrt[3]{6}\left(\text{cis}\left(\frac{25\pi}{18}\right)\right)}$$

75. Find all complex cube roots of $z = 4 + 4i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 4\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

The formula for the modulus and argument of the cube roots are

$$|\sqrt[3]{z}| = \sqrt[3]{|z|}, \qquad \arg\left(\sqrt[3]{z}\right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.$$

So for this case,

$$|\sqrt[3]{z}| = \sqrt[3]{4\sqrt{2}} = \sqrt[3]{4}\sqrt[6]{2}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{4} + 2k\pi}{3}, \quad k = 0, 1, 2.$$

$$k = 0: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{4}\sqrt[6]{2}\left(\text{cis}\left(\frac{\frac{\pi}{4} + 0}{3}\right)\right) = \boxed{\sqrt[3]{4}\sqrt[6]{2}\left(\text{cis}\left(\frac{\pi}{12}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{4}\sqrt[6]{2}\left(\text{cis}\left(\frac{\frac{\pi}{4} + \frac{8\pi}{4}}{3}\right)\right) = \boxed{\sqrt[3]{4}\sqrt[6]{2}\left(\text{cis}\left(\frac{9\pi}{12}\right)\right)}$$

$$k = 2: \quad \Rightarrow \quad \sqrt[3]{z} = \sqrt[3]{4}\sqrt[6]{2}\left(\text{cis}\left(\frac{\frac{\pi}{4} + \frac{16\pi}{4}}{3}\right)\right) = \boxed{\sqrt[3]{4}\sqrt[6]{2}\left(\text{cis}\left(\frac{17\pi}{12}\right)\right)}$$

76. Find all complex square roots of $z = \sqrt{3} + i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{2}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{6} + 2k\pi}{2}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt{z} = \sqrt{2}\left(\text{cis}\left(\frac{\frac{\pi}{6} + 0}{2}\right)\right) = \boxed{\sqrt{2}\left(\text{cis}\left(\frac{\pi}{12}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt{z} = \sqrt{2}\left(\text{cis}\left(\frac{\frac{\pi}{6} + \frac{12\pi}{6}}{2}\right)\right) = \boxed{\sqrt{2}\left(\text{cis}\left(\frac{13\pi}{12}\right)\right)}$$

77. Find all complex square roots of $z = 4\sqrt{3} + 4i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 8\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{8}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{6} + 2k\pi}{2}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt{z} = \sqrt{8}\left(\operatorname{cis}\left(\frac{\frac{\pi}{6} + 0}{2}\right)\right) = \boxed{\sqrt{8}\left(\operatorname{cis}\left(\frac{\pi}{12}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt{z} = \sqrt{8}\left(\operatorname{cis}\left(\frac{\frac{\pi}{6} + \frac{12\pi}{6}}{2}\right)\right) = \boxed{\sqrt{8}\left(\operatorname{cis}\left(\frac{13\pi}{12}\right)\right)}$$

78. Find all complex square roots of $z = 3 + 3i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 3\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{3\sqrt{2}} = \sqrt{3}\sqrt[4]{2}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{4} + 2k\pi}{2}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt{z} = \sqrt{3}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\frac{\pi}{4} + 0}{2}\right)\right) = \boxed{\sqrt{3}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\pi}{8}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt{z} = \sqrt{3}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{\frac{\pi}{4} + \frac{8\pi}{4}}{2}\right)\right) = \boxed{\sqrt{3}\sqrt[4]{2}\left(\operatorname{cis}\left(\frac{9\pi}{8}\right)\right)}$$

79. Find all complex square roots of $z = 3 + 3\sqrt{3}i$.

Solution. Note that z is in Quadrant 1. Its polar form is

$$z = |z|(\cos\theta + i\sin\theta) = 6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \qquad \arg\left(\sqrt{z}\right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$

So for this case,

$$|\sqrt{z}| = \sqrt{6}, \qquad \arg\left(\sqrt{z}\right) = \frac{\frac{\pi}{3} + 2k\pi}{2}, \quad k = 0, 1.$$

$$k = 0: \quad \Rightarrow \quad \sqrt{z} = \sqrt{6}\left(\text{cis}\left(\frac{\frac{\pi}{3} + 0}{2}\right)\right) = \boxed{\sqrt{6}\left(\text{cis}\left(\frac{\pi}{6}\right)\right)}$$

$$k = 1: \quad \Rightarrow \quad \sqrt{z} = \sqrt{6}\left(\text{cis}\left(\frac{\frac{\pi}{3} + \frac{6\pi}{3}}{2}\right)\right) = \boxed{\sqrt{6}\left(\text{cis}\left(\frac{7\pi}{6}\right)\right)}$$

80. Find the cube roots of unity.

Solution. The formula for complex cube roots is

$$\sqrt[3]{z} = \sqrt[3]{|z|}\text{cis}\left(\frac{\arg(z) + 2k\pi}{3}\right), \quad k = 0, 1, 2.$$

Roots of unity are the roots of 1, and so using $|1| = 1$ and $\arg(1) = 0$, this reduces to the roots of unity formula

$$\sqrt[3]{1} = \text{cis}\left(\frac{2k\pi}{3}\right), \quad k = 0, 1, 2.$$

Specifying these to each k, we have

$$k = 0: \quad \sqrt[3]{1} = \boxed{\text{cis}\,(0) = 1}$$

$$k = 1: \quad \sqrt[3]{1} = \boxed{\text{cis}\left(\frac{2\pi}{3}\right)}$$

$$k = 2: \quad \sqrt[3]{1} = \boxed{\text{cis}\left(\frac{4\pi}{3}\right)}$$

In rectangular coordinates, these are

$$\boxed{\sqrt[3]{1} = 1, \quad \sqrt[3]{1} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \sqrt[3]{1} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i}.$$

81. Find the fourth roots of unity.

Solution. The formula for complex fourth roots is

$$\sqrt[4]{z} = \sqrt[4]{|z|}\text{cis}\left(\frac{\arg(z) + 2k\pi}{4}\right), \quad k = 0, 1, 2, 3.$$

Roots of unity are the roots of 1, and so using $|1| = 1$ and $\arg(1) = 0$, this reduces to the roots of unity formula

$$\sqrt[4]{1} = \text{cis}\left(\frac{2k\pi}{4}\right), \quad k = 0, 1, 2, 3.$$

Specifying these to each k, we have

$$k = 0: \quad \sqrt[4]{1} = \boxed{\text{cis}(0)}$$

$$k = 1: \quad \sqrt[4]{1} = \boxed{\text{cis}\left(\frac{\pi}{2}\right)}$$

$$k = 2: \quad \sqrt[4]{1} = \boxed{\text{cis}(\pi)}$$

$$k = 3: \quad \sqrt[4]{1} = \boxed{\text{cis}\left(\frac{3\pi}{2}\right)}$$

In rectangular coordinates, these are

$$\boxed{\sqrt[4]{1} = 1, \quad \sqrt[4]{1} = i, \quad \sqrt[4]{1} = -1, \quad \sqrt[4]{1} = -i}.$$

82. Find the fifth roots of unity.

Solution. The formula for complex fifth roots is

$$\sqrt[5]{z} = \sqrt[5]{|z|}\text{cis}\left(\frac{\arg(z) + 2k\pi}{5}\right), \quad k = 0, 1, 2, 3, 4.$$

Roots of unity are the roots of 1, and so using $|1| = 1$ and $\arg(1) = 0$, this reduces to the roots of unity formula

$$\sqrt[5]{1} = \text{cis}\left(\frac{2k\pi}{5}\right), \quad k = 0, 1, 2, 3, 4.$$

Specifying these to each k, we have

$$k = 0: \quad \sqrt[5]{1} = \boxed{\text{cis}(0)} \qquad k = 1: \quad \sqrt[5]{1} = \boxed{\text{cis}\left(\frac{2\pi}{5}\right)}$$

$$k = 2: \quad \sqrt[5]{1} = \boxed{\text{cis}\left(\frac{4\pi}{5}\right)} \qquad k = 3: \quad \sqrt[5]{1} = \boxed{\text{cis}\left(\frac{6\pi}{5}\right)}$$

$$k = 4: \quad \sqrt[5]{1} = \boxed{\text{cis}\left(\frac{8\pi}{5}\right)}$$

13.5

1. Compute the complex exponential $\exp(z)$ for $z = 6 - 3i$.

   Solution. The formula for the complex exponential is

   $$\exp(x + iy) = e^x (\cos(y) + i \sin(y)).$$

   Here, $x = 6$ and $y = -3$, so we have

   $$\exp(6 - 3i) = e^6 (\cos(-3) + i \sin(-3)).$$

2. Compute the complex exponential $\exp(z)$ for $z = -1 + 2i$.

   Solution. The formula for the complex exponential is

   $$\exp(x + iy) = e^x (\cos(y) + i \sin(y)).$$

   Here, $x = -1$ and $y = 2$, so we have

   $$\exp(-1 + 2i) = e^{-1} (\cos(2) + i \sin(2)).$$

3. Compute the complex exponential $\exp(z)$ for $z = 5 + 11i$.

   Solution. The formula for the complex exponential is

   $$\exp(x + iy) = e^x (\cos(y) + i \sin(y)).$$

   Here, $x = 5$ and $y = 11$, so we have

   $$\exp(5 + 11i) = e^5 (\cos(11) + i \sin(11)).$$

4. For the complex number $z = x + iy$, find the real and imaginary parts of the quantity $\exp(z^2)$ in terms of x and y.

Solution. We compute

$$z^2 = (x+iy)(x+iy) = (x^2-y^2) + (2xy)i,$$

and using the formula for the complex exponential

$$\exp(x+iy) = e^x(\cos y + i\sin y),$$

$$\exp(z^2) = \exp(x^2-y^2)\left(\cos(2xy) + i\sin(2xy)\right).$$

The real part is the term with no i, and the imaginary part is the coefficient of i, so they are, resp.

$$\boxed{\exp(x^2-y^2)\cos(2xy), \quad \exp(x^2-y^2)\sin(2xy)}$$

5. Find all solutions to the equation

$$\exp(z) = -3+i.$$

Solution. We find a number z such that when the complex exponential of z is computed, we get a number with modulus and argument the same as $-3+i$.

$$|-3+i| = \sqrt{(-3)^2+(1)^2} = \sqrt{10}, \quad \arg(-3+i) = \arctan\left(-\frac{1}{3}\right) + \pi$$

(the arctangent is not exact because z is in quadrant 2). From the formula

$$\exp(x+iy) = e^x\left(\cos(y) + i\sin(y)\right),$$

$$|\exp(z)| = e^x, \qquad \text{and} \qquad \arg(\exp(z)) = y + 2k\pi, \quad k\in\mathbb{Z}.$$

$$e^x = \sqrt{10}, \quad \Rightarrow \quad x = \ln\left(\sqrt{10}\right),$$

$$y + 2k\pi = \arctan\left(-\frac{1}{3}\right) + \pi, \quad \Rightarrow \quad y = \arctan\left(-\frac{1}{3}\right) + \pi + 2k\pi, \quad k\in\mathbb{Z},$$

and so the solutions are

$$z = x+iy = \boxed{\ln\left(\sqrt{10}\right) + \left(\arctan\left(-\frac{1}{3}\right) + \pi + 2k\pi\right)i, \quad k\in\mathbb{Z}}.$$

One could also reduce the terms $\pi + 2k\pi$, $k\in\mathbb{Z}$ to $(2k+1)\pi$, $k\in\mathbb{Z}$.

6. Find all solutions to the equation

$$\exp(z) = 3 + i.$$

Solution. We find a number z such that when the complex exponential of z is computed, we get a number with modulus and argument the same as $3 + i$.

$$|3 + i| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}, \quad \arg(3 + i) = \arctan\left(\frac{1}{3}\right)$$

(the arctangent is exact because z is in quadrant 1). From the formula

$$\exp(x + iy) = e^x (\cos(y) + i\sin(y)),$$

we have

$$|\exp(z)| = e^x, \qquad \text{and} \qquad \arg(\exp(z)) = y + 2k\pi, \quad k \in \mathbb{Z}.$$

So

$$e^x = \sqrt{10}, \quad \Rightarrow \quad x = \ln\left(\sqrt{10}\right),$$

$$y + 2k\pi = \arctan\left(\frac{1}{3}\right), \quad \Rightarrow \quad y = \arctan\left(\frac{1}{3}\right) + 2k\pi, \quad k \in \mathbb{Z},$$

and so the solutions are

$$z = x + iy = \boxed{\ln\left(\sqrt{10}\right) + \left(\arctan\left(\frac{1}{3}\right) + 2k\pi\right)i, \quad k \in \mathbb{Z}}.$$

7. Find all solutions to the equation

$$\exp(z) = 1 + 2i.$$

Solution. We find a number z such that when the complex exponential of z is computed, we get a number with modulus and argument the same as $1 + 2i$.

$$|1 + 2i| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}, \quad \arg(1 + 2i) = \arctan(2)$$

(the arctangent is exact because z is in quadrant 1). From the formula

$$\exp(x + iy) = e^x (\cos(y) + i\sin(y)),$$

we have

$$|\exp(z)| = e^x, \qquad \text{and} \qquad \arg(\exp(z)) = y + 2k\pi, \quad k \in \mathbb{Z}.$$

So

$$e^x = \sqrt{5}, \quad \Rightarrow \quad x = \ln\left(\sqrt{5}\right),$$

$$y + 2k\pi = \arctan(2), \quad \Rightarrow \quad y = \arctan(2) + 2k\pi, \quad k \in \mathbb{Z},$$

and so the solutions are

$$z = x + iy = \boxed{\ln\left(\sqrt{5}\right) + (\arctan(2) + 2k\pi)\,i, \quad k \in \mathbb{Z}}.$$

8. Find all solutions to the equation

$$\exp(z) = 2 + 3i.$$

Solution. We find a number z such that when the complex exponential of z is computed, we get a number with modulus and argument the same as $2 + 3i$.

$$|2+3i| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}, \quad \arg(2+3i) = \arctan\left(\frac{3}{2}\right)$$

(the arctangent is exact because z is in quadrant 1). From the formula

$$\exp(x+iy) = e^x (\cos(y) + i\sin(y)),$$

we have

$$|\exp(z)| = e^x, \qquad \text{and} \qquad \arg(\exp(z)) = y + 2k\pi, \quad k \in \mathbb{Z}.$$

So

$$e^x = \sqrt{13}, \quad \Rightarrow \quad x = \ln\left(\sqrt{13}\right),$$

$$y + 2k\pi = \arctan\left(\frac{3}{2}\right), \quad \Rightarrow \quad y = \arctan\left(\frac{3}{2}\right) + 2k\pi, \quad k \in \mathbb{Z},$$

and so the solutions are

$$z = x + iy = \boxed{\ln\left(\sqrt{13}\right) + \left(\arctan\left(\frac{3}{2}\right) + 2k\pi\right) i, \quad k \in \mathbb{Z}}.$$

9. Find all solutions to the equation

$$\exp(z) = -1 + 3i.$$

Solution. We find a number z such that when the complex exponential of z is computed, we get a number with modulus and argument the same as $-1 + 3i$.

$$|-1+3i| = \sqrt{(-1)^2 + (3)^2} = \sqrt{10}, \quad \arg(-1+3i) = \arctan(-3) + \pi$$

(the arctangent is not exact because z is in quadrant 2). From the formula

$$\exp(x+iy) = e^x (\cos(y) + i\sin(y)),$$

$$|\exp(z)| = e^x, \qquad \text{and} \qquad \arg(\exp(z)) = y + 2k\pi, \quad k \in \mathbb{Z}.$$

$$e^x = \sqrt{10}, \quad \Rightarrow \quad x = \ln\left(\sqrt{10}\right),$$

$$y + 2k\pi = \arctan(-3) + \pi, \quad \Rightarrow \quad y = \arctan(-3) + \pi + 2k\pi, \quad k \in \mathbb{Z},$$

and so the solutions are

$$z = x + iy = \boxed{\ln\left(\sqrt{10}\right) + (\arctan(-3) + \pi + 2k\pi)\, i, \quad k \in \mathbb{Z}}.$$

One could also reduce the terms $\pi + 2k\pi$, $k \in \mathbb{Z}$ to $(2k+1)\pi$, $k \in \mathbb{Z}$.

10. Find all solutions to the equation

$$\exp(z) = -1 + 2i.$$

Solution. We find a number z such that when the complex exponential of z is computed, we get a number with modulus and argument the same as $-1+2i$.

$$|-1+2i| = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}, \quad \arg(-1+2i) = \arctan(-2) + \pi$$

(the arctangent is not exact because z is in quadrant 2). From the formula

$$\exp(x+iy) = e^x (\cos(y) + i\sin(y)),$$

$$|\exp(z)| = e^x, \qquad \text{and} \qquad \arg(\exp(z)) = y + 2k\pi, \quad k \in \mathbb{Z}.$$

$$e^x = \sqrt{5}, \quad \Rightarrow \quad x = \ln\left(\sqrt{5}\right),$$

$$y + 2k\pi = \arctan(-2) + \pi, \quad \Rightarrow \quad y = \arctan(-2) + \pi + 2k\pi, \quad k \in \mathbb{Z},$$

and so the solutions are

$$z = x + iy = \boxed{\ln\left(\sqrt{5}\right) + (\arctan(-2) + \pi + 2k\pi)\, i, \quad k \in \mathbb{Z}}.$$

One could also reduce the terms $\pi + 2k\pi$, $k \in \mathbb{Z}$ to $(2k+1)\pi$, $k \in \mathbb{Z}$.

13.7

1. Compute all values of $\ln(1+2i)$.

Solution. We first find the polar form of $z = 1 + 2i$.

$$|1+2i| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}, \quad \arg(1+2i) = \arctan(2)$$

(the arctangent is exact because z is in quadrant 1). From the formula

$$\ln(re^{i\theta}) = \ln(r) + i\theta,$$

we have

$$\ln(1+2i) = \boxed{\ln\left(\sqrt{5}\right) + (\arctan(2) + 2k\pi)\, i, \quad k \in \mathbb{Z}}.$$

2. Compute all values of $\ln(-3+i)$.

Solution. We first find the polar form of $z = -3+i$.

$$|3+i| = \sqrt{(3)^2+(1)^2} = \sqrt{10}, \quad \arg(-3+i) = \arctan\left(-\frac{1}{3}+\pi\right)$$

(the arctangent is off by π because z is in quadrant 2). From the formula

$$\ln(re^{i\theta}) = \ln(r) + i\theta,$$

we have

$$\ln(3+i) = \boxed{\ln\left(\sqrt{10}\right) + \left(\arctan\left(-\frac{1}{3}\right)+\pi+2k\pi\right)i, \quad k\in\mathbb{Z}}.$$

3. Compute all values of $\ln(-2+i)$.

Solution. We first find the polar form of $z = -2+i$.

$$|2+i| = \sqrt{(2)^2+(1)^2} = \sqrt{5}, \quad \arg(-2+i) = \arctan\left(-\frac{1}{2}+\pi\right)$$

(the arctangent is off by π because z is in quadrant 2). From the formula

$$\ln(re^{i\theta}) = \ln(r) + i\theta,$$

we have

$$\ln(2+i) = \boxed{\ln\left(\sqrt{5}\right) + \left(\arctan\left(-\frac{1}{2}\right)+\pi+2k\pi\right)i, \quad k\in\mathbb{Z}}.$$

4. Compute all values of $\ln(2+3i)$.

Solution. We first find the polar form of $z = 2 + 3i$.

$$|2+3i| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}, \quad \arg(2+3i) = \arctan\left(\frac{3}{2}\right)$$

(the arctangent is exact because z is in quadrant 1). From the formula

$$\ln(re^{i\theta}) = \ln(r) + i\theta,$$

we have

$$\ln(2+3i) = \boxed{\ln\left(\sqrt{13}\right) + \left(\arctan\left(\frac{3}{2}\right) + 2k\pi\right)i, \quad k \in \mathbb{Z}}.$$

5. Compute all values of $\ln(3+i)$.

Solution. We first find the polar form of $z = 3 + i$.

$$|3+i| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}, \quad \arg(3+i) = \arctan\left(\frac{1}{3}\right)$$

(the arctangent is exact because z is in quadrant 1). From the formula

$$\ln(re^{i\theta}) = \ln(r) + i\theta,$$

we have

$$\ln(3+i) = \boxed{\ln\left(\sqrt{10}\right) + \left(\arctan\left(\frac{1}{3}\right) + 2k\pi\right)i, \quad k \in \mathbb{Z}}.$$

6. Compute all values of $\ln(-3+2i)$.

Solution. We first find the polar form of $z = -3 + 2i$.

$$|3+2i| = \sqrt{(3)^2 + (2)^2} = \sqrt{13}, \quad \arg(-3+2i) = \arctan\left(-\frac{2}{3} + \pi\right)$$

(the arctangent is off by π because z is in quadrant 2). From the formula

$$\ln(re^{i\theta}) = \ln(r) + i\theta,$$

we have

$$\ln(3+2i) = \boxed{\ln\left(\sqrt{13}\right) + \left(\arctan\left(-\frac{2}{3}\right) + \pi + 2k\pi\right) i, \quad k \in \mathbb{Z}}.$$

7. Find all values of $(-3+3i)^{4i}$ (that's the $4i$ power of $-3+3i$, not multiplication).

Solution. The formula for general powers is

$$z^c = \exp(c\ln(z)).$$

To find $\ln(z)$, we compute

$$|z| = \sqrt{(-3)^2 + (3)^2} = \sqrt{18}, \quad \arg(z) = \frac{3}{4}\pi + 2k\pi, \quad k \in \mathbb{Z},$$

$$\ln(z) = \ln\left(\sqrt{18}\right) + i\left(\frac{3}{4}\pi + 2k\pi\right), \quad k \in \mathbb{Z}.$$

$$c\ln(z) = (-3\pi + 2k\pi) + i\left(4\ln\left(\sqrt{18}\right)\right), \quad k \in \mathbb{Z}.$$

Finally,

$$z^c = \exp(c\ln(z)) = \boxed{\exp(-3\pi + 2k\pi)\left(\operatorname{cis}\left(4\ln\sqrt{18}\right)\right), \quad k \in \mathbb{Z}.}$$

8. Find all values of $(-4+4i)^{2i}$ (that's the $2i$ power of $-4+4i$, not multiplication).

Solution. The formula for general powers is

$$z^c = \exp(c\ln(z)).$$

To find $\ln(z)$, we compute

$$|z| = \sqrt{(-4)^2 + (4)^2} = \sqrt{32}, \quad \arg(z) = \frac{3}{4}\pi + 2k\pi, \quad k \in \mathbb{Z},$$

$$\ln(z) = \ln\left(\sqrt{32}\right) + i\left(\frac{3}{4}\pi + 2k\pi\right), \quad k \in \mathbb{Z}.$$

$$c\ln(z) = \left(-\frac{3}{2}\pi + 2k\pi\right) + i\left(2\ln\left(\sqrt{32}\right)\right), \quad k \in \mathbb{Z}.$$

Finally,

$$z^c = \exp(c\ln(z)) = \boxed{\exp\left(-\frac{3}{2}\pi + 2k\pi\right)\left(\text{cis}\left(2\ln\sqrt{32}\right)\right), \quad k \in \mathbb{Z}.}$$

9. Find all values of $(-3+3i)^{2i}$ (that's the $2i$ power of $-3+3i$, not multiplication).

Solution. The formula for general powers is

$$z^c = \exp(c\ln(z)).$$

To find $\ln(z)$, we compute

$$|z| = \sqrt{(-3)^2 + (3)^2} = \sqrt{18}, \quad \arg(z) = \frac{3}{4}\pi + 2k\pi, \quad k \in \mathbb{Z},$$

$$\ln(z) = \ln\left(\sqrt{18}\right) + i\left(\frac{3}{4}\pi + 2k\pi\right), \quad k \in \mathbb{Z}.$$

$$c\ln(z) = \left(-\frac{3}{2}\pi + 2k\pi\right) + i\left(2\ln\left(\sqrt{18}\right)\right), \quad k \in \mathbb{Z}.$$

Finally,

$$z^c = \exp(c\ln(z)) = \boxed{\exp\left(-\frac{3}{2}\pi + 2k\pi\right)\left(\text{cis}\left(2\ln\sqrt{18}\right)\right), \quad k \in \mathbb{Z}.}$$

10. Find all values of $(-4+4i)^{3i}$ (that's the $3i$ power of $-4+4i$, not multiplication).

Solution. The formula for general powers is

$$z^c = \exp(c\ln(z)).$$

To find $\ln(z)$, we compute

$$|z| = \sqrt{(-4)^2+(4)^2} = \sqrt{32}, \quad \arg(z) = \frac{3}{4}\pi + 2k\pi, \quad k \in \mathbb{Z},$$

$$\ln(z) = \ln\left(\sqrt{32}\right) + i\left(\frac{3}{4}\pi + 2k\pi\right), \quad k \in \mathbb{Z}.$$

$$c\ln(z) = \left(-\frac{9}{4}\pi + 2k\pi\right) + i\left(3\ln\left(\sqrt{32}\right)\right), \quad k \in \mathbb{Z}.$$

Finally,

$$z^c = \exp(c\ln(z)) = \boxed{\exp\left(-\frac{9}{4}\pi + 2k\pi\right)\left(\operatorname{cis}\left(3\ln\sqrt{32}\right)\right), \quad k \in \mathbb{Z}.}$$

11. Find all values of $(-5+5i)^{2i}$ (that's the $2i$ power of $-5+5i$, not multiplication).

Solution. The formula for general powers is

$$z^c = \exp(c\ln(z)).$$

To find $\ln(z)$, we compute

$$|z| = \sqrt{(-5)^2+(5)^2} = \sqrt{50}, \quad \arg(z) = \frac{3}{4}\pi + 2k\pi, \quad k \in \mathbb{Z},$$

$$\ln(z) = \ln\left(\sqrt{50}\right) + i\left(\frac{3}{4}\pi + 2k\pi\right), \quad k \in \mathbb{Z}.$$

$$c\ln(z) = \left(-\frac{3}{2}\pi + 2k\pi\right) + i\left(2\ln\left(\sqrt{50}\right)\right), \quad k \in \mathbb{Z}.$$

Finally,

$$z^c = \exp(c\ln(z)) = \boxed{\exp\left(-\frac{3}{2}\pi + 2k\pi\right)\left(\operatorname{cis}\left(2\ln\sqrt{50}\right)\right), \quad k \in \mathbb{Z}.}$$