13.1

1. For $z_1 = -1 + 2i$ and $z_2 = 3 + i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

   Solution.

   \[
   z_1 + z_2 = (-1 + 2i) + (3 + i) = (2 + 3i)
   \]

   \[
   z_1 - z_2 = (-1 + 2i) - (3 + i) = (-4 + i)
   \]

   \[
   z_1 z_2 = (-1 + 2i)(3 + i) = -5 + 5i
   \]

   \[
   \frac{z_1}{z_2} = \frac{-1 + 2i}{3 + i} = \frac{-1 + 7i}{10} = \frac{-1}{10} + \frac{7}{10}i
   \]

   \[
   \frac{1}{z_2} = \frac{3 - i}{(3 + i)(3 - i)} = \frac{3 - i}{10} = \frac{3}{10} - \frac{1}{10}i
   \]

2. For $z_1 = -2 - 3i$ and $z_2 = -2 + 5i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

   Solution.

   \[
   z_1 + z_2 = (-2 - 3i) + (-2 + 5i) = (-2 - 2) + (-3 + 5)i = -4 + 2i
   \]

   \[
   z_1 - z_2 = (-2 - 3i) - (-2 + 5i) = (-2 + 2) + (-3 - 5)i = -8i
   \]

   \[
   z_1 z_2 = (-2 - 3i)(-2 + 5i) = 4 - 15i^2 - 10i + 6i = 19 - 4i
   \]

   \[
   \frac{z_1}{z_2} = \frac{-2 - 3i}{-2 + 5i} = \frac{-2 - 5i}{(-2 + 5i)(-2 - 5i)} = \frac{-11 + 16i}{29} = \frac{-11}{29} + \frac{16}{29}i
   \]

   \[
   \frac{1}{z_2} = \frac{1}{-2 + 5i} = \frac{-2 - 5i}{(-2 + 5i)(-2 - 5i)} = \frac{-2 - 5i}{29} = \frac{-2}{29} - \frac{5}{29}i
   \]

3. For $z_1 = -3 + 3i$ and $z_2 = 3 + 6i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, $\frac{z_1}{z_2}$, and $\frac{1}{z_2}$.

   Solution.

   \[
   z_1 + z_2 = (-3 + 3i) + (3 + 6i) = (3 + 3) + (3 + 6)i = 6i
   \]

   \[
   z_1 - z_2 = (-3 + 3i) - (3 + 6i) = (-3 - 3) + (3 - 6)i = -6 - 3i
   \]

   \[
   z_1 z_2 = (-3 + 3i)(3 + 6i) = 9 + 18i^2 - 18i + 9i = -27 - 9i
   \]

   \[
   \frac{z_1}{z_2} = \frac{-3 + 3i}{3 + 6i} = \frac{-3 + 3i}{(3 + 6i)(3 - 6i)} = \frac{9 + 27i}{45} = \frac{1}{5} + \frac{3}{5}i
   \]

   \[
   \frac{1}{z_2} = \frac{1}{3 + 6i} = \frac{3 - 6i}{(3 + 6i)(3 - 6i)} = \frac{3 - 6i}{45} = \frac{1}{15} - \frac{2}{15}i
   \]
4. For \( z_1 = +4i \) and \( z_2 = -3 + 2i \), find \( z_1 + z_2 \), \( z_1 - z_2 \), \( z_1z_2 \), \( \frac{z_1}{z_2} \), and \( \frac{1}{z_2} \).

**Solution.**

\[
\begin{align*}
z_1 + z_2 &= (+4i) + (-3 + 2i) = (0 - 3) + (4 + 2)i = -3 + 6i \\
z_1 - z_2 &= (+4i) - (-3 + 2i) = (0 + 3) + (4 - 2)i = 3 + 2i \\
z_1z_2 &= (+4i)(-3 + 2i) = 0 + 8i^2 + 0i - 12i = -8 - 12i \\
\frac{z_1}{z_2} &= \frac{+4i}{-3 + 2i} = \frac{(4i)(-3 - 2i)}{(-3 + 2i)(-3 - 2i)} = \frac{8 - 12i}{-9 + 4} = \frac{8 - 12i}{13} \\
\frac{1}{z_2} &= \frac{1}{-3 + 2i} = \frac{-3 - 2i}{(-3 + 2i)(-3 - 2i)} = \frac{-3 - 2i}{9 + 4} = \frac{-3 - 2i}{13}
\end{align*}
\]

5. For \( z_1 = -4 - i \) and \( z_2 = 2 + 3i \), find \( z_1 + z_2 \), \( z_1 - z_2 \), \( z_1z_2 \), \( \frac{z_1}{z_2} \), and \( \frac{1}{z_2} \).

**Solution.**

\[
\begin{align*}
z_1 + z_2 &= (-4 - i) + (2 + 3i) = (-4 + 2) + (-1 + 3)i = -2 + 2i \\
z_1 - z_2 &= (-4 - i) - (2 + 3i) = (-4 - 2) + (-1 - 3)i = -6 - 4i \\
z_1z_2 &= (-4 - i)(2 + 3i) = -8 - 3i^2 - 12i - 2i = -5 - 14i \\
\frac{z_1}{z_2} &= \frac{-4 - i}{2 + 3i} = \frac{(-4 - i)(2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{-11 + 10i}{13} = \frac{-11 + 10i}{13} \\
\frac{1}{z_2} &= \frac{1}{2 + 3i} = \frac{2 - 3i}{(2 + 3i)(2 - 3i)} = \frac{2 - 3i}{13} = \frac{2 - 3i}{13}
\end{align*}
\]

6. For \( z_1 = 5 + 2i \) and \( z_2 = +i \), find \( z_1 + z_2 \), \( z_1 - z_2 \), \( z_1z_2 \), \( \frac{z_1}{z_2} \), and \( \frac{1}{z_2} \).

**Solution.**

\[
\begin{align*}
z_1 + z_2 &= (5 + 2i) + (+i) = (5 + 0) + (2 + 1)i = 5 + 3i \\
z_1 - z_2 &= (5 + 2i) - (+i) = (5 + 0) + (2 - 1)i = 5 + i \\
z_1z_2 &= (5 + 2i)(+i) = 0 + 2i^2 + 5i + 0i = -2 + 5i \\
\frac{z_1}{z_2} &= \frac{5 + 2i}{+i} = \frac{(5 + 2i)(-i)}{(-i)(-i)} = \frac{2 - 5i}{1} = 2 - 5i \\
\frac{1}{z_2} &= \frac{1}{+i} = \frac{-i}{(-i)(-i)} = \frac{-i}{1} = 0 - 1i
\end{align*}
\]

7. For a given complex number \( z = x + iy \), find the real and imaginary parts of \( z^2 \) and \( 1/z \).
Solution. To find the real or imaginary part of a quantity, first compute the quantity and work through it so that there is exactly one $i$ in your answer; that is, write it in the form $x + iy$. Then pull out $x$ (for the real part) or $y$ (for the imaginary part).

$$z^2 = (x + iy)^2 = x^2 + 2xyi - y^2 = \frac{x^2 - y^2}{\text{Re}(z^2)} + i \frac{2xy}{\text{Im}(z^2)}$$

So $\text{Re}(z^2) = x^2 - y^2$ and $\text{Im}(z^2) = 2xy$ (note that $i$ is not in the imaginary part).

$$\frac{1}{z} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

So $\text{Re}(1/z) = \frac{x}{x^2 + y^2}$ and $\text{Im}(1/z) = \frac{-y}{x^2 + y^2}$ (don’t forget the negative sign).

8. Find the modulus of the complex number $z = 1 + i$.

Solution.

$$|z| = |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$  

9. Find the modulus of the complex number $z = -2 + 3i$.

Solution.

$$|z| = |-2 + 3i| = \sqrt{-2^2 + 3^2} = \sqrt{13}.$$  

10. Find the modulus of the complex number $z = -9 + i$.

Solution.

$$|z| = |-9 + i| = \sqrt{-9^2 + 1^2} = \sqrt{82}.$$  

11. Find the modulus of the complex number $z = -5 + 4i$.

Solution.

$$|z| = |-5 + 4i| = \sqrt{-5^2 + 4^2} = \sqrt{41}.$$  

12. Find the modulus of the complex number $z = -5 - 3i$.  

13.2

1. Convert the complex number \( z = -2 + 2i \) to polar form.

\[ y \quad 2 \quad -2 \quad \sqrt{2}/2 \quad -\sqrt{2}/2, \]
we see that
\[ \sin \theta = \frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{-\sqrt{2}}{2}, \]
and therefore \( \theta = \frac{3\pi}{4} \). For the modulus, we compute
\[ |z| = \sqrt{(-2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}, \]
and so the polar form is
\[ z = |z|(\cos \theta + i \sin \theta) = 2\sqrt{2} \left( \cos \left( \frac{3\pi}{4} \right) + i \sin \left( \frac{3\pi}{4} \right) \right). \]

2. Convert the complex number \( z = -4 - 4i \) to polar form.

\[ y \quad -4 \quad -4 \quad -\sqrt{2}/2 \quad -\sqrt{2}/2, \]
we see that
\[ \sin \theta = -\frac{\sqrt{2}}{2}, \quad \cos \theta = -\frac{\sqrt{2}}{2}, \]
and therefore \( \theta = \frac{5\pi}{4} \). For the modulus, we compute
\[ |z| = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}, \]
and so the polar form is
\[ z = |z|(\cos \theta + i \sin \theta) = 4\sqrt{2} \left( \cos \left( \frac{5\pi}{4} \right) + i \sin \left( \frac{5\pi}{4} \right) \right). \]

3. Convert the complex number \( z = 2\sqrt{3} - 2i \) to polar form.
Solution. Note that $z$ is in Quadrant 4. From the ratio
\[
\frac{y}{x} = \frac{-2}{2\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2},
\]
we see that
\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = -\frac{\sqrt{3}}{2},
\]
and therefore $\theta = -\frac{\pi}{6}$. For the modulus, we compute
\[
|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4,
\]
and so the polar form is
\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right).
\]

4. Convert the complex number $z = -1 - i$ to polar form.

Solution. Note that $z$ is in Quadrant 3. From the ratio
\[
\frac{y}{x} = \frac{-1}{-1} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2},
\]
we see that
\[
\sin \theta = -\frac{\sqrt{2}}{2}, \quad \cos \theta = -\frac{\sqrt{2}}{2},
\]
and therefore $\theta = \frac{5\pi}{4}$. For the modulus, we compute
\[
|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} = \sqrt{2},
\]
and so the polar form is
\[
z = |z|(\cos \theta + i \sin \theta) = \sqrt{2} \left( \cos \left( \frac{5\pi}{4} \right) + i \sin \left( \frac{5\pi}{4} \right) \right).
\]

5. Convert the complex number $z = 4i$ to polar form.

Solution. Note that $z$ is pure imaginary (real part is zero), and therefore its argument is always $\pm \frac{\pi}{2}$. Since $4 > 0$, $\theta = \frac{\pi}{2}$.
For the modulus, we compute
\[
|z| = \sqrt{(0)^2 + (4)^2} = \sqrt{16} = 4,
\]
and so the polar form is
\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right).
\]
6. Convert the complex number \( z = -3i \) to polar form.

**Solution.** Note that \( z \) is pure imaginary (real part is zero), and therefore its argument is always \( \pm \frac{\pi}{2} \). Since \(-3 < 0\), \( \theta = -\frac{\pi}{2} \).

For the modulus, we compute

\[
|z| = \sqrt{(0)^2 + (-3)^2} = \sqrt{9} = 3,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 3 \left( \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right).
\]

7. Convert the complex number \( z = -2i \) to polar form.

**Solution.** Note that \( z \) is pure imaginary (real part is zero), and therefore its argument is always \( \pm \frac{\pi}{2} \). Since \(-2 < 0\), \( \theta = -\frac{\pi}{2} \).

For the modulus, we compute

\[
|z| = \sqrt{(0)^2 + (-2)^2} = \sqrt{4} = 2,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 2 \left( \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right).
\]

8. Convert the complex number \( z = 2 - 3i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 4. To find the modulus, we consider the ratio

\[
\frac{y}{x} = \frac{-3}{2} = -\frac{3}{2},
\]

and so we seek an angle \( \theta \) such that the numerator is (a multiple of) \( \sin \theta \) and the denominator is (a multiple of) \( \cos \theta \). There certainly exists such an angle, but it is not among our memorized values on the unit circle. Therefore, the best that we can do is

\[
\theta = \arctan \left( -\frac{3}{2} \right).
\]

This value is exact since \( z \) is in either Quadrant 1 or 4.

For the modulus, we compute

\[
|z| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13},
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = \sqrt{13} \left( \cos \left( \arctan \left( -\frac{3}{2} \right) \right) + i \sin \left( \arctan \left( -\frac{3}{2} \right) \right) \right).
\]

9. Convert the complex number \( z = 1 - i \) to polar form.
Solution. Note that $z$ is in Quadrant 4. From the ratio
\[ \frac{y}{x} = \frac{-1}{1} = -\frac{\sqrt{2}}{2}, \]
we see that
\[ \sin \theta = -\frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{\sqrt{2}}{2}, \]
and therefore $\theta = \frac{7\pi}{4}$. For the modulus, we compute
\[ |z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} = \sqrt{2}, \]
and so the polar form is
\[ z = |z|(\cos \theta + i \sin \theta) = \sqrt{2} \left( \cos \left( \frac{7\pi}{4} \right) + i \sin \left( \frac{7\pi}{4} \right) \right). \]

10. Convert the complex number $z = 1 + i$ to polar form.

Solution. Note that $z$ is in Quadrant 1. From the ratio
\[ \frac{y}{x} = \frac{1}{1} = \frac{\sqrt{2}}{2}, \]
we see that
\[ \sin \theta = \frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{\sqrt{2}}{2}, \]
and therefore $\theta = \frac{\pi}{4}$. For the modulus, we compute
\[ |z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} = \sqrt{2}, \]
and so the polar form is
\[ z = |z|(\cos \theta + i \sin \theta) = \sqrt{2} \left( \cos \left( \frac{\pi}{4} \right) + i \sin \left( \frac{\pi}{4} \right) \right). \]

11. Convert the complex number $z = 5\sqrt{3} + 5i$ to polar form.

Solution. Note that $z$ is in Quadrant 1. From the ratio
\[ \frac{y}{x} = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}/2}, \]
we see that
\[ \sin \theta = \frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2}, \]
and therefore $\theta = \frac{\pi}{6}$. For the modulus, we compute
\[ |z| = \sqrt{(5\sqrt{3})^2 + (5)^2} = \sqrt{100} = 10, \]
and so the polar form is
\[ z = |z|(\cos \theta + i \sin \theta) = 10 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right). \]
12. Convert the complex number \( z = -\sqrt{3} + i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 2. From the ratio

\[
\frac{y}{x} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{1}{2}/\sqrt{3/2},
\]

we see that

\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2},
\]

and therefore \( \theta = \frac{5\pi}{6} \). For the modulus, we compute

\[
|z| = \sqrt{(-1\sqrt{3})^2 + 1^2} = \sqrt{4} = 2,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 2 \left( \cos \left( \frac{5\pi}{6} \right) + i \sin \left( \frac{5\pi}{6} \right) \right).
\]

13. Convert the complex number \( z = -1 + i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 2. From the ratio

\[
\frac{y}{x} = \frac{1}{-1} = \frac{\sqrt{2}/2}{-\sqrt{2}/2},
\]

we see that

\[
\sin \theta = \frac{\sqrt{2}}{2}, \quad \cos \theta = -\frac{\sqrt{2}}{2},
\]

and therefore \( \theta = \frac{3\pi}{4} \). For the modulus, we compute

\[
|z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} = \sqrt{2},
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = \sqrt{2} \left( \cos \left( \frac{3\pi}{4} \right) + i \sin \left( \frac{3\pi}{4} \right) \right).
\]

14. Convert the complex number \( z = 2 + 0i \) to polar form.

**Solution.** Note that \( z \) is actually a real number (imaginary part is zero), and therefore its argument is always either 0 or \( \pi \). Since \( 2 > 0 \), \( \theta = 0 \).

For the modulus, we compute

\[
|z| = \sqrt{(2)^2 + (0)^2} = \sqrt{4} = 2,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 2 \left( \cos (0) + i \sin (0) \right).
\]
15. Convert the complex number \( z = 2 + 3i \) to polar form.

\textit{Solution.} Note that \( z \) is in Quadrant 1. To find the modulus, we consider the ratio

\[
\frac{y}{x} = \frac{3}{2} = \frac{3}{2},
\]

and so we seek an angle \( \theta \) such that the numerator is (a multiple of) \( \sin \theta \) and the denominator is (a multiple of) \( \cos \theta \). There certainly exists such an angle, but it is not among our memorized values on the unit circle. Therefore, the best that we can do is

\[
\theta = \arctan \left( \frac{3}{2} \right).
\]

This value is exact since \( z \) is in either Quadrant 1 or 4.

For the modulus, we compute

\[
|z| = \sqrt{(2)^2 + (3)^2} = \sqrt{13},
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = \sqrt{13} \left( \cos \left( \arctan \left( \frac{3}{2} \right) \right) + i \sin \left( \arctan \left( \frac{3}{2} \right) \right) \right).\]

16. Convert the complex number \( z = 4 + 0i \) to polar form.

\textit{Solution.} Note that \( z \) is actually a real number (imaginary part is zero), and therefore its argument is always either 0 or \( \pi \). Since \( 4 > 0 \), \( \theta = 0 \).

For the modulus, we compute

\[
|z| = \sqrt{(4)^2 + (0)^2} = \sqrt{16} = 4,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos (0) + i \sin (0) \right).
\]

17. Convert the complex number \( z = 2i \) to polar form.

\textit{Solution.} Note that \( z \) is pure imaginary (real part is zero), and therefore its argument is always \( \pm \frac{\pi}{2} \). Since \( 2 > 0 \), \( \theta = \frac{\pi}{2} \).

For the modulus, we compute

\[
|z| = \sqrt{(0)^2 + (2)^2} = \sqrt{4} = 2,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 2 \left( \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right).\]
18. Convert the complex number \( z = -5\sqrt{3} + 5i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 2. From the ratio

\[
\frac{y}{x} = \frac{5}{-5\sqrt{3}} = \frac{-1}{\sqrt{3}} = -\frac{1}{2}\sqrt{3}/2,
\]

we see that

\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2},
\]

and therefore \( \theta = \frac{5\pi}{6} \). For the modulus, we compute

\[
|z| = \sqrt{(-5\sqrt{3})^2 + 5^2} = \sqrt{100} = 10,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 10 \left( \cos \left( \frac{5\pi}{6} \right) + i \sin \left( \frac{5\pi}{6} \right) \right).
\]

19. Convert the complex number \( z = \sqrt{3} - i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 4. From the ratio

\[
\frac{y}{x} = \frac{-1}{1\sqrt{3}} = \frac{-1}{\sqrt{3}} = -\frac{1}{2}\sqrt{3}/2,
\]

we see that

\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2},
\]

and therefore \( \theta = -\frac{\pi}{6} \). For the modulus, we compute

\[
|z| = \sqrt{(1\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 2 \left( \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right).
\]

20. Convert the complex number \( z = -2 + 0i \) to polar form.

**Solution.** Note that \( z \) is actually a real number (imaginary part is zero), and therefore its argument is always either 0 or \( \pi \). Since \(-2 < 0\), \( \theta = \pi \).

For the modulus, we compute

\[
|z| = \sqrt{(-2)^2 + (0)^2} = \sqrt{4} = 2,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 2 \left( \cos \left( \pi \right) + i \sin \left( \pi \right) \right).
\]
21. Convert the complex number \( z = -2\sqrt{3} - 2i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 3. From the ratio

\[
\frac{y}{x} = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{-1/2}{-\sqrt{3}/2},
\]

we see that

\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = -\frac{\sqrt{3}}{2},
\]

and therefore \( \theta = -\frac{5\pi}{6} \). For the modulus, we compute

\[
|z| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos \left( -\frac{5\pi}{6} \right) + i \sin \left( -\frac{5\pi}{6} \right) \right).
\]

22. Convert the complex number \( z = -4 + 4i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 2. From the ratio

\[
\frac{y}{x} = \frac{4}{-4} = -\frac{\sqrt{2}/2}{\sqrt{2}/2},
\]

we see that

\[
\sin \theta = \frac{\sqrt{2}}{2}, \quad \cos \theta = -\frac{\sqrt{2}}{2},
\]

and therefore \( \theta = \frac{3\pi}{4} \). For the modulus, we compute

\[
|z| = \sqrt{(-4)^2 + (4)^2} = \sqrt{32} = 4\sqrt{2},
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 4\sqrt{2} \left( \cos \left( \frac{3\pi}{4} \right) + i \sin \left( \frac{3\pi}{4} \right) \right).
\]

23. Convert the complex number \( z = 3 + 3i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 1. From the ratio

\[
\frac{y}{x} = \frac{3}{3} = \frac{\sqrt{2}/2}{\sqrt{2}/2},
\]

we see that

\[
\sin \theta = \frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{\sqrt{2}}{2},
\]

and therefore \( \theta = \frac{\pi}{4} \). For the modulus, we compute

\[
|z| = \sqrt{(3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2},
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 3\sqrt{2} \left( \cos \left( \frac{\pi}{4} \right) + i \sin \left( \frac{\pi}{4} \right) \right).
\]
24. Convert the complex number \( z = -2 - 2\sqrt{3}i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 3. From the ratio

\[
\frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \frac{-\sqrt{3}}{-1} = \frac{-\sqrt{3}/2}{-1/2},
\]

we see that

\[
\sin \theta = -\frac{\sqrt{3}}{2}, \quad \cos \theta = -\frac{1}{2},
\]

and therefore \( \theta = \frac{4\pi}{3} \). For the modulus, we compute

\[
|z| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos \left( \frac{4\pi}{3} \right) + i \sin \left( \frac{4\pi}{3} \right) \right).
\]

25. Convert the complex number \( z = -3\sqrt{3} - 3i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 3. From the ratio

\[
\frac{y}{x} = \frac{-3}{-3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1/2}{-\sqrt{3}/2},
\]

we see that

\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = -\frac{\sqrt{3}}{2},
\]

and therefore \( \theta = -\frac{5\pi}{6} \). For the modulus, we compute

\[
|z| = \sqrt{(-3\sqrt{3})^2 + (-3)^2} = \sqrt{36} = 6,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 6 \left( \cos \left( -\frac{5\pi}{6} \right) + i \sin \left( -\frac{5\pi}{6} \right) \right).
\]

26. Convert the complex number \( z = 3\sqrt{3} - 3i \) to polar form.

**Solution.** Note that \( z \) is in Quadrant 4. From the ratio

\[
\frac{y}{x} = \frac{-3}{3\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2},
\]

we see that

\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2},
\]

and therefore \( \theta = -\frac{\pi}{6} \). For the modulus, we compute

\[
|z| = \sqrt{(3\sqrt{3})^2 + (-3)^2} = \sqrt{36} = 6,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 6 \left( \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right).
\]
27. Convert the complex number \( z = 2 - 2\sqrt{3}i \) to polar form.

\textbf{Solution.} Note that \( z \) is in Quadrant 4. From the ratio
\[
\frac{y}{x} = \frac{-2\sqrt{3}}{2} = -\frac{\sqrt{3}}{1} = -\frac{\sqrt{3}/2}{1/2},
\]
we see that
\[
\sin \theta = -\frac{\sqrt{3}}{2}, \quad \cos \theta = \frac{1}{2},
\]
and therefore \( \theta = -\frac{\pi}{3} \). For the modulus, we compute
\[
|z| = \sqrt{(-2\sqrt{3})^2 + (2)^2} = \sqrt{16} = 4,
\]
and so the polar form is
\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos \left(-\frac{\pi}{3}\right) + i \sin \left(-\frac{\pi}{3}\right) \right).
\]

28. Convert the complex number \( z = 2 + 2\sqrt{3}i \) to polar form.

\textbf{Solution.} Note that \( z \) is in Quadrant 1. From the ratio
\[
\frac{y}{x} = \frac{2\sqrt{3}}{2} = \frac{\sqrt{3}}{1} = \frac{\sqrt{3}/2}{1/2},
\]
we see that
\[
\sin \theta = \frac{\sqrt{3}}{2}, \quad \cos \theta = \frac{1}{2},
\]
and therefore \( \theta = \frac{\pi}{3} \). For the modulus, we compute
\[
|z| = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{16} = 4,
\]
and so the polar form is
\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos \left(\frac{\pi}{3}\right) + i \sin \left(\frac{\pi}{3}\right) \right).
\]

29. Convert the complex number \( z = -2 - 3i \) to polar form.
Solution. Note that \( z \) is in Quadrant 3. To find the modulus, we consider the ratio

\[
\frac{y}{x} = \frac{-3}{-2} = \frac{3}{2},
\]

and so we seek an angle \( \theta \) such that the numerator is (a multiple of) \( \sin \theta \) and the denominator is (a multiple of) \( \cos \theta \). There certainly exists such an angle, but it is not among our memorized values on the unit circle. Therefore, we consider \( \arctan \left( \frac{3}{2} \right) \). Since the range of the arctangent function is only in quadrants 1 and 4, we need to add \( \pi \) to this value to get the true argument of \( z = -2 - 3i \), so we have

\[
\theta = \arctan \left( \frac{3}{2} \right) + \pi.
\]

Remember this goes outside of the arctangent function. For the modulus, we compute

\[
|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13},
\]

and so the polar form is

\[
z = \sqrt{13} \left( \cos \left( \arctan \left( \frac{3}{2} \right) + \pi \right) + i \sin \left( \arctan \left( \frac{3}{2} \right) + \pi \right) \right).
\]

30. Convert the complex number \( z = -4\sqrt{3} + 4i \) to polar form.

Solution. Note that \( z \) is in Quadrant 2. From the ratio

\[
\frac{y}{x} = \frac{4}{-4\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2},
\]

we see that

\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2},
\]

and therefore \( \theta = \frac{5\pi}{6} \). For the modulus, we compute

\[
|z| = \sqrt{(-4\sqrt{3})^2 + 4^2} = \sqrt{64} = 8,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 8 \left( \cos \left( \frac{5\pi}{6} \right) + i \sin \left( \frac{5\pi}{6} \right) \right).
\]

31. Convert the complex number \( z = 4\sqrt{3} - 4i \) to polar form.
**Solution.** Note that $z$ is in Quadrant 4. From the ratio
$$\frac{y}{x} = \frac{-4}{4\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-1/2}{\sqrt{3}/2},$$
we see that
$$\sin \theta = -\frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2},$$
and therefore $\theta = -\frac{\pi}{6}$. For the modulus, we compute
$$|z| = \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{64} = 8,$$
and so the polar form is
$$z = |z|(\cos \theta + i \sin \theta) = 8 \left( \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right).$$

32. Convert the complex number $z = \sqrt{3} + i$ to polar form.

**Solution.** Note that $z$ is in Quadrant 1. From the ratio
$$\frac{y}{x} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1/2}{\sqrt{3}/2},$$
we see that
$$\sin \theta = \frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2},$$
and therefore $\theta = \frac{\pi}{6}$. For the modulus, we compute
$$|z| = \sqrt{(1\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2,$$
and so the polar form is
$$z = |z|(\cos \theta + i \sin \theta) = 2 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right).$$

33. Convert the complex number $z = 5 + 5i$ to polar form.

**Solution.** Note that $z$ is in Quadrant 1. From the ratio
$$\frac{y}{x} = \frac{5}{5} = \frac{\sqrt{2}/2}{\sqrt{2}/2},$$
we see that
$$\sin \theta = \frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{\sqrt{2}}{2},$$
and therefore $\theta = \frac{\pi}{4}$. For the modulus, we compute
$$|z| = \sqrt{(5)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2},$$
and so the polar form is
$$z = |z|(\cos \theta + i \sin \theta) = 5\sqrt{2} \left( \cos \left( \frac{\pi}{4} \right) + i \sin \left( \frac{\pi}{4} \right) \right).$$
34. Convert the complex number \( z = -2 + 2\sqrt{3}i \) to polar form.

*Solution.* Note that \( z \) is in Quadrant 2. From the ratio
\[
\frac{y}{x} = -\frac{2\sqrt{3}}{2} = -\frac{\sqrt{3}}{1} = -\sqrt{3}/2,
\]
we see that
\[
\sin \theta = -\frac{\sqrt{3}}{2}, \quad \cos \theta = -\frac{1}{2},
\]
and therefore \( \theta = \frac{2\pi}{3} \). For the modulus, we compute
\[
|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4,
\]
and so the polar form is
\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right).
\]

35. Convert the complex number \( z = -\sqrt{3} - i \) to polar form.

*Solution.* Note that \( z \) is in Quadrant 3. From the ratio
\[
\frac{y}{x} = -\frac{-1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{1/2}{\sqrt{3}/2},
\]
we see that
\[
\sin \theta = -\frac{1}{2}, \quad \cos \theta = -\frac{\sqrt{3}}{2},
\]
and therefore \( \theta = -\frac{5\pi}{6} \). For the modulus, we compute
\[
|z| = \sqrt{(-1\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2,
\]
and so the polar form is
\[
z = |z|(\cos \theta + i \sin \theta) = 2 \left( \cos \left( -\frac{5\pi}{6} \right) + i \sin \left( -\frac{5\pi}{6} \right) \right).
\]

36. Convert the complex number \( z = -2 + 3i \) to polar form.
Solution. Note that \( z \) is in Quadrant 2. To find the modulus, we consider the ratio

\[
\frac{y}{x} = \frac{3}{-2} = -\frac{3}{2},
\]

and so we seek an angle \( \theta \) such that the numerator is (a multiple of) \( \sin \theta \) and the denominator is (a multiple of) \( \cos \theta \). There certainly exists such an angle, but it is not among our memorized values on the unit circle. Therefore, we consider \( \arctan \left( -\frac{3}{2} \right) \).

Since the range of the arctangent function is only in quadrants 1 and 4, we need to add \( \pi \) to this value to get the true argument of \( z = -2 + 3i \), so we have

\[
\theta = \arctan \left( -\frac{3}{2} \right) + \pi.
\]

Remember this goes outside of the arctangent function. For the modulus, we compute

\[
|z| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13},
\]

and so the polar form is

\[
z = \sqrt{13} \left( \cos \left( \arctan \left( -\frac{3}{2} \right) + \pi \right) + i \sin \left( \arctan \left( -\frac{3}{2} \right) + \pi \right) \right).
\]

37. Convert the complex number \( z = -3 + 0i \) to polar form.

Solution. Note that \( z \) is actually a real number (imaginary part is zero), and therefore its argument is always either 0 or \( \pi \). Since \(-3 < 0\), \( \theta = \pi \).

For the modulus, we compute

\[
|z| = \sqrt{(-3)^2 + (0)^2} = \sqrt{9} = 3,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 3(\cos(\pi) + i \sin(\pi)).
\]

38. Convert the complex number \( z = 2\sqrt{3} + 2i \) to polar form.

Solution. Note that \( z \) is in Quadrant 1. From the ratio

\[
\frac{y}{x} = \frac{2\sqrt{3}}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1/2}{\sqrt{3}/2},
\]

we see that

\[
\sin \theta = \frac{1}{2}, \quad \cos \theta = \frac{\sqrt{3}}{2},
\]

and therefore \( \theta = \frac{\pi}{6} \). For the modulus, we compute

\[
|z| = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{16} = 4,
\]

and so the polar form is

\[
z = |z|(\cos \theta + i \sin \theta) = 4 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right).
\]
39. Convert the complex number $z = 2 - 2i$ to polar form.

**Solution.** Note that $z$ is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-2}{2} = -\frac{\sqrt{2}}{2},$$

we see that

$$\sin \theta = -\frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{7\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2},$$

and so the polar form is

$$z = |z|(\cos \theta + i \sin \theta) = 2\sqrt{2} \left( \cos \left( \frac{7\pi}{4} \right) + i \sin \left( \frac{7\pi}{4} \right) \right).$$

40. Convert the complex number $z = 5 - 5i$ to polar form.

**Solution.** Note that $z$ is in Quadrant 4. From the ratio

$$\frac{y}{x} = \frac{-5}{5} = -\frac{\sqrt{2}}{2},$$

we see that

$$\sin \theta = -\frac{\sqrt{2}}{2}, \quad \cos \theta = \frac{\sqrt{2}}{2},$$

and therefore $\theta = \frac{7\pi}{4}$. For the modulus, we compute

$$|z| = \sqrt{(5)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2},$$

and so the polar form is

$$z = |z|(\cos \theta + i \sin \theta) = 5\sqrt{2} \left( \cos \left( \frac{7\pi}{4} \right) + i \sin \left( \frac{7\pi}{4} \right) \right).$$

41. For the complex numbers $z_1 = 2 \left( \cos \left( \frac{1}{7}\pi \right) + i \sin \left( \frac{1}{7}\pi \right) \right)$ and $z_2 = 3 \left( \cos \left( \frac{2}{5}\pi \right) + i \sin \left( \frac{2}{5}\pi \right) \right)$, compute $z_1z_2$, $\frac{z_1}{z_2}$, $z_1^2$, and $z_2^3$.

**Solution.**

$$z_1z_2 = (2)(3) \text{cis} \left( \frac{1}{7}\pi + \frac{2}{5}\pi \right) = 6\text{cis} \left( \frac{19}{35}\pi \right)$$

$$\frac{z_1}{z_2} = \frac{2}{3} \text{cis} \left( \frac{1}{7}\pi - \frac{2}{5}\pi \right) = \frac{2}{3} \text{cis} \left( -\frac{9}{35}\pi \right)$$

$$z_1^2 = (2^2) \text{cis} \left( 2 \cdot \frac{1}{7}\pi \right) = 4\text{cis} \left( \frac{2}{7}\pi \right)$$

$$z_2^3 = (3^3) \text{cis} \left( 3 \cdot \frac{2}{5}\pi \right) = 27\text{cis} \left( \frac{6}{5}\pi \right)$$
42. For the complex numbers \( z_1 = 2 \left( \cos \left( \frac{-2}{7} \pi \right) + i \sin \left( \frac{-2}{7} \pi \right) \right) \) and \( z_2 = 2 \left( \cos \left( \frac{2}{11} \pi \right) + i \sin \left( \frac{2}{11} \pi \right) \right) \), compute \( z_1 z_2 \), \( \frac{z_1}{z_2} \), \( z_1^5 \), and \( z_2^4 \).

**Solution.**

\[
z_1 z_2 = (2) (2) \text{ cis } \left( \frac{-2}{7} \pi + \frac{2}{11} \pi \right) = 4 \text{ cis } \left( \frac{-8}{77} \pi \right)
\]

\[
\frac{z_1}{z_2} = \frac{2}{2} \text{ cis } \left( \frac{-2}{7} \pi - \frac{2}{11} \pi \right) = 1 \text{ cis } \left( \frac{-36}{77} \pi \right)
\]

\[
z_1^5 = (2^5) \text{ cis } \left( 5 \frac{-2}{7} \pi \right) = 32 \text{ cis } \left( \frac{-10}{7} \pi \right)
\]

\[
z_2^4 = (2^4) \text{ cis } \left( 4 \frac{2}{11} \pi \right) = 16 \text{ cis } \left( \frac{8}{11} \pi \right)
\]

43. For the complex numbers \( z_1 = 3 \left( \cos \left( \frac{-1}{11} \pi \right) + i \sin \left( \frac{-1}{11} \pi \right) \right) \) and \( z_2 = 2 \left( \cos \left( \frac{5}{13} \pi \right) + i \sin \left( \frac{5}{13} \pi \right) \right) \), compute \( z_1 z_2 \), \( \frac{z_1}{z_2} \), \( z_1^2 \), and \( z_2^4 \).

**Solution.**

\[
z_1 z_2 = (3) (2) \text{ cis } \left( \frac{-1}{11} \pi + \frac{5}{13} \pi \right) = 6 \text{ cis } \left( \frac{42}{143} \pi \right)
\]

\[
\frac{z_1}{z_2} = \frac{3}{2} \text{ cis } \left( \frac{-1}{11} \pi - \frac{5}{13} \pi \right) = \frac{3}{2} \text{ cis } \left( \frac{-68}{143} \pi \right)
\]

\[
z_1^2 = (3^2) \text{ cis } \left( 2 \frac{-1}{11} \pi \right) = 9 \text{ cis } \left( \frac{-2}{11} \pi \right)
\]

\[
z_2^4 = (2^4) \text{ cis } \left( 4 \frac{5}{13} \pi \right) = 16 \text{ cis } \left( \frac{20}{13} \pi \right)
\]

44. For the complex numbers \( z_1 = 2 \left( \cos \left( \frac{4}{5} \pi \right) + i \sin \left( \frac{4}{5} \pi \right) \right) \) and \( z_2 = \left( \cos \left( \frac{3}{7} \pi \right) + i \sin \left( \frac{3}{7} \pi \right) \right) \), compute \( z_1 z_2 \), \( \frac{z_1}{z_2} \), \( z_1^4 \), and \( z_2^3 \).

**Solution.**

\[
z_1 z_2 = (2) (1) \text{ cis } \left( \frac{4}{5} \pi + \frac{3}{7} \pi \right) = 2 \text{ cis } \left( \frac{43}{35} \pi \right)
\]

\[
\frac{z_1}{z_2} = \frac{2}{1} \text{ cis } \left( \frac{4}{5} \pi - \frac{3}{7} \pi \right) = 2 \text{ cis } \left( \frac{13}{35} \pi \right)
\]

\[
z_1^4 = (2^4) \text{ cis } \left( 4 \frac{4}{5} \pi \right) = 16 \text{ cis } \left( \frac{16}{5} \pi \right)
\]

\[
z_2^3 = (1^3) \text{ cis } \left( 3 \frac{3}{7} \pi \right) = \text{ cis } \left( \frac{9}{7} \pi \right)
\]
45. For the complex numbers \( z_1 = \left( \cos \left( -\frac{3}{11} \pi \right) + i \sin \left( -\frac{3}{11} \pi \right) \right) \) and \( z_2 = \left( \cos \left( -\frac{3}{7} \pi \right) + i \sin \left( -\frac{3}{7} \pi \right) \right) \), compute \( z_1 z_2 \), \( \frac{z_1}{z_2} \), \( z_1^2 \), and \( z_2^4 \).

**Solution.**

\[
z_1 z_2 = (1)(1) \ cis \left( -\frac{3}{11} \pi + -\frac{3}{7} \pi \right) = \ cis \left( -\frac{54}{77} \pi \right)
\]

\[
\frac{z_1}{z_2} = \frac{1}{1} \ cis \left( -\frac{3}{11} \pi - -\frac{3}{7} \pi \right) = 1 \ cis \left( \frac{12}{77} \pi \right)
\]

\[
z_1^2 = (1^2) \ cis \left( 2 \left( -\frac{3}{11} \pi \right) \right) = \ cis \left( -\frac{6}{11} \pi \right)
\]

\[
z_2^4 = (1^4) \ cis \left( 4 \left( -\frac{3}{7} \pi \right) \right) = \ cis \left( -\frac{12}{7} \pi \right)
\]

46. Find all cube roots of \( z = 2 \left( \cos \left( \frac{3}{10} \pi \right) + i \sin \left( \frac{3}{10} \pi \right) \right) \). You may leave your answer in polar form.

**Solution.** The formula for the modulus and argument of the cube roots are

\[
|\sqrt[3]{z}| = \sqrt[3]{|z|}, \quad \arg \left( \sqrt[3]{z} \right) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.
\]

So for this case, since \( z \) is given in polar form,

\[
|\sqrt[3]{z}| = \sqrt[3]{2}, \quad \arg \left( \sqrt[3]{z} \right) = \frac{3\pi}{10} + \frac{2k\pi}{3}, \quad k = 0, 1, 2.
\]

Then for each value of \( k \),

\[
k = 0: \quad \arg \left( \sqrt[3]{z} \right) = \frac{3\pi}{10} = \frac{1}{10} \pi \quad \Rightarrow \sqrt[3]{z} = \sqrt[3]{2} \left( \cis \left( \frac{1}{10} \pi \right) \right)
\]

\[
k = 1: \quad \arg \left( \sqrt[3]{z} \right) = \frac{3\pi}{10} + \frac{2\pi}{3} = \frac{23}{30} \pi \quad \Rightarrow \sqrt[3]{z} = \sqrt[3]{2} \left( \cis \left( \frac{23}{30} \pi \right) \right)
\]

\[
k = 2: \quad \arg \left( \sqrt[3]{z} \right) = \frac{3\pi}{10} + \frac{4\pi}{3} = \frac{43}{30} \pi \quad \Rightarrow \sqrt[3]{z} = \sqrt[3]{2} \left( \cis \left( \frac{43}{30} \pi \right) \right)
\]

47. Find all cube roots of \( z = 2 \left( \cos \left( \frac{3}{11} \pi \right) + i \sin \left( \frac{3}{11} \pi \right) \right) \). You may leave your answer in polar form.
Solution. The formula for the modulus and argument of the cube roots are

\[
|\sqrt[3]{z}| = \sqrt[3]{|z|}, \quad \arg(\sqrt[3]{z}) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.
\]

So for this case, since \(z\) is given in polar form,

\[
|\sqrt[3]{z}| = \sqrt[3]{2}, \quad \arg(\sqrt[3]{z}) = \frac{\pi}{3} + \frac{2k\pi}{3}, \quad k = 0, 1, 2.
\]

Then for each value of \(k\),

- \(k = 0: \arg(\sqrt[3]{z}) = \frac{\pi}{3} = \frac{1}{11}\pi\)
  \[
  \sqrt[3]{z} = \sqrt[3]{2} \left(\text{cis} \left(\frac{\pi}{11}\right)\right)
  \]

- \(k = 1: \arg(\sqrt[3]{z}) = \frac{\pi}{3} + \frac{2\pi}{3} = \frac{25}{33}\pi\)
  \[
  \sqrt[3]{z} = \sqrt[3]{2} \left(\text{cis} \left(\frac{25}{33}\pi\right)\right)
  \]

- \(k = 2: \arg(\sqrt[3]{z}) = \frac{\pi}{3} + \frac{4\pi}{3} = \frac{47}{33}\pi\)
  \[
  \sqrt[3]{z} = \sqrt[3]{2} \left(\text{cis} \left(\frac{47}{33}\pi\right)\right)
  \]

48. Find all cube roots of \(z = 2 \left(\cos \left(\frac{2}{7}\pi\right) + i \sin \left(\frac{2}{7}\pi\right)\right)\). You may leave your answer in polar form.

Solution. The formula for the modulus and argument of the cube roots are

\[
|\sqrt[3]{z}| = \sqrt[3]{|z|}, \quad \arg(\sqrt[3]{z}) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.
\]

So for this case, since \(z\) is given in polar form,

\[
|\sqrt[3]{z}| = \sqrt[3]{2}, \quad \arg(\sqrt[3]{z}) = \frac{2\pi}{3} + \frac{2k\pi}{3}, \quad k = 0, 1, 2.
\]

Then for each value of \(k\),

- \(k = 0: \arg(\sqrt[3]{z}) = \frac{2\pi}{3} = \frac{21}{21}\pi\)
  \[
  \sqrt[3]{z} = \sqrt[3]{2} \left(\text{cis} \left(\frac{2}{21}\pi\right)\right)
  \]

- \(k = 1: \arg(\sqrt[3]{z}) = \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{16}{21}\pi\)
  \[
  \sqrt[3]{z} = \sqrt[3]{2} \left(\text{cis} \left(\frac{16}{21}\pi\right)\right)
  \]

- \(k = 2: \arg(\sqrt[3]{z}) = \frac{2\pi}{3} + \frac{4\pi}{3} = \frac{10}{7}\pi\)
  \[
  \sqrt[3]{z} = \sqrt[3]{2} \left(\text{cis} \left(\frac{10}{7}\pi\right)\right)
  \]
49. Find all cube roots of \( z = 3 \left( \cos \left( \frac{3}{5} \pi \right) + i \sin \left( \frac{3}{5} \pi \right) \right) \). You may leave your answer in polar form.

\textbf{Solution.} The formula for the modulus and argument of the cube roots are 
\[
| \sqrt[3]{z} | = \sqrt[3]{|z|}, \quad \arg ( \sqrt[3]{z} ) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.
\]
So for this case, since \( z \) is given in polar form, 
\[
| \sqrt[3]{z} | = \sqrt[3]{3}, \quad \arg ( \sqrt[3]{z} ) = \frac{3\pi + 2k\pi}{3}, \quad k = 0, 1, 2.
\]
Then for each value of \( k \),
\[
k = 0: \quad \arg ( \sqrt[3]{z} ) = \frac{3\pi}{3} = \frac{1}{5} \pi \quad \sqrt[3]{z} = \sqrt[3]{3} \left( \text{cis} \left( \frac{1}{5} \pi \right) \right)
\]
\[
k = 1: \quad \arg ( \sqrt[3]{z} ) = \frac{3\pi + 2\pi}{3} = \frac{13}{15} \pi \quad \sqrt[3]{z} = \sqrt[3]{3} \left( \text{cis} \left( \frac{13}{15} \pi \right) \right)
\]
\[
k = 2: \quad \arg ( \sqrt[3]{z} ) = \frac{3\pi + 4\pi}{3} = \frac{23}{15} \pi \quad \sqrt[3]{z} = \sqrt[3]{3} \left( \text{cis} \left( \frac{23}{15} \pi \right) \right)
\]

50. Find all cube roots of \( z = 2 \left( \cos \left( \frac{3}{7} \pi \right) + i \sin \left( \frac{3}{7} \pi \right) \right) \). You may leave your answer in polar form.

\textbf{Solution.} The formula for the modulus and argument of the cube roots are 
\[
| \sqrt[3]{z} | = \sqrt[3]{|z|}, \quad \arg ( \sqrt[3]{z} ) = \frac{\arg(z) + 2k\pi}{3}, \quad k = 0, 1, 2.
\]
So for this case, since \( z \) is given in polar form, 
\[
| \sqrt[3]{z} | = \sqrt[3]{2}, \quad \arg ( \sqrt[3]{z} ) = \frac{3\pi + 2k\pi}{3}, \quad k = 0, 1, 2.
\]
Then for each value of \( k \),
\[
k = 0: \quad \arg ( \sqrt[3]{z} ) = \frac{3\pi}{3} = \frac{1}{7} \pi \quad \sqrt[3]{z} = \sqrt[3]{2} \left( \text{cis} \left( \frac{1}{7} \pi \right) \right)
\]
\[
k = 1: \quad \arg ( \sqrt[3]{z} ) = \frac{3\pi + 2\pi}{3} = \frac{17}{21} \pi \quad \sqrt[3]{z} = \sqrt[3]{2} \left( \text{cis} \left( \frac{17}{21} \pi \right) \right)
\]
\[
k = 2: \quad \arg ( \sqrt[3]{z} ) = \frac{3\pi + 4\pi}{3} = \frac{31}{21} \pi \quad \sqrt[3]{z} = \sqrt[3]{2} \left( \text{cis} \left( \frac{31}{21} \pi \right) \right)
\]
51. Find all square roots of \( z = -1 - 4i \).

**Solution.** We first convert to polar coordinates. In Quadrant 3, the polar form of \( z \) is

\[
z = \sqrt{17} \left( \text{cis} \left( \arctan(4) + \pi \right) \right)
\]

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of \( \pm \pi \) to get the argument for \( z \). The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{17} = \sqrt{17}, \quad \arg(\sqrt{z}) = \frac{\arctan(4) + \pi + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt{17} \left( \text{cis} \left( \frac{\arctan(4) + \pi}{2} \right) \right)
\]

\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt{17} \left( \text{cis} \left( \frac{\arctan(4) + 3\pi}{2} \right) \right)
\]

52. Find all square roots of \( z = 3 - i \).

**Solution.** We first convert to polar coordinates. In Quadrant 4, the polar form of \( z \) is

\[
z = \sqrt{10} \left( \cos \left( \arctan\left( -\frac{1}{3} \right) \right) + i \sin \left( \arctan\left( -\frac{1}{3} \right) \right) \right)
\]

The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{10} = \sqrt{10}, \quad \arg(\sqrt{z}) = \frac{\arctan(-1/3) + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt{10} \left( \text{cis} \left( \frac{\arctan(-1/3)}{2} \right) \right)
\]

\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt{10} \left( \text{cis} \left( \frac{\arctan(-1/3) + 2\pi}{2} \right) \right)
\]
53. Find all square roots of \( z = 1 + 5i \).

**Solution.** We first convert to polar coordinates. In Quadrant 1, the polar form of \( z \) is

\[
z = \sqrt{26} \left( \cos \left( \arctan(5) \right) + i \sin \left( \arctan(5) \right) \right)
\]

The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = |z|, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{26} = \sqrt{26}, \quad \arg(\sqrt{z}) = \frac{\arctan(5) + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \implies \sqrt{z} = \sqrt[4]{26} \left( \cis \left( \frac{\arctan(5)}{2} \right) \right)
\]

\[
k = 1 \implies \sqrt{z} = \sqrt[4]{26} \left( \cis \left( \frac{\arctan(5) + 2\pi}{2} \right) \right)
\]

54. Find all square roots of \( z = 4 + i \).

**Solution.** We first convert to polar coordinates. In Quadrant 1, the polar form of \( z \) is

\[
z = \sqrt{17} \left( \cos \left( \arctan(1/4) \right) + i \sin \left( \arctan(1/4) \right) \right)
\]

The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = |z|, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{17} = \sqrt{17}, \quad \arg(\sqrt{z}) = \frac{\arctan(1/4) + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \implies \sqrt{z} = \sqrt[4]{17} \left( \cis \left( \frac{\arctan(1/4)}{2} \right) \right)
\]

\[
k = 1 \implies \sqrt{z} = \sqrt[4]{17} \left( \cis \left( \frac{\arctan(1/4) + 2\pi}{2} \right) \right)
\]
55. Find all square roots of \( z = -5 - i \).

**Solution.** We first convert to polar coordinates. In Quadrant 3, the polar form of \( z \) is

\[
z = \sqrt{26} \left( \cos(\arctan(1/5) + \pi) + i \sin(\arctan(1/5) + \pi) \right)
\]

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of \( \pm \pi \) to get the argument for \( z \). The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{26}, \quad \arg(\sqrt{z}) = \frac{\arctan(1/5) + \pi + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \implies \sqrt{z} = \sqrt[4]{26} \left( \cos \left( \frac{\arctan(1/5) + \pi}{2} \right) + i \sin \left( \frac{\arctan(1/5) + \pi}{2} \right) \right)
\]

\[
k = 1 \implies \sqrt{z} = \sqrt[4]{26} \left( \cos \left( \frac{\arctan(1/5) + 3\pi}{2} \right) + i \sin \left( \frac{\arctan(1/5) + 3\pi}{2} \right) \right)
\]

56. Find all square roots of \( z = 2 - 5i \).

**Solution.** We first convert to polar coordinates. In Quadrant 4, the polar form of \( z \) is

\[
z = \sqrt{29} \left( \cos(\arctan(-5/2)) + i \sin(\arctan(-5/2)) \right)
\]

The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{29}, \quad \arg(\sqrt{z}) = \frac{\arctan(-5/2) + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \implies \sqrt{z} = \sqrt{29} \left( \cos \left( \frac{\arctan(-5/2)}{2} \right) + i \sin \left( \frac{\arctan(-5/2)}{2} \right) \right)
\]

\[
k = 1 \implies \sqrt{z} = \sqrt[4]{29} \left( \cos \left( \frac{\arctan(-5/2) + 2\pi}{2} \right) + i \sin \left( \frac{\arctan(-5/2) + 2\pi}{2} \right) \right)
\]
57. Find all square roots of \( z = -2 + 3i \).

**Solution.** We first convert to polar coordinates. In Quadrant 2, the polar form of \( z \) is

\[
z = \sqrt{13} \left( \text{cis} \left( \arctan \left( \frac{-3}{2} \right) + \pi \right) \right)
\]

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of \( \pm \pi \) to get the argument for \( z \). The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \text{arg} (\sqrt{z}) = \frac{\text{arg}(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{\sqrt{13}} = \sqrt{13}, \quad \text{arg} (\sqrt{z}) = \frac{\arctan(-3/2) + \pi + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt[4]{13} \left( \text{cis} \left( \arctan \left( \frac{-3}{2} \right) + \pi \right) \right)
\]

\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt[4]{13} \left( \text{cis} \left( \arctan \left( \frac{-3}{2} \right) + 3\pi \right) \right)
\]

58. Find all square roots of \( z = -3 + i \).

**Solution.** We first convert to polar coordinates. In Quadrant 2, the polar form of \( z \) is

\[
z = \sqrt{10} \left( \text{cis} \left( \arctan \left( \frac{-1}{3} \right) + \pi \right) \right)
\]

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of \( \pm \pi \) to get the argument for \( z \). The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \text{arg} (\sqrt{z}) = \frac{\text{arg}(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{\sqrt{10}} = \sqrt{10}, \quad \text{arg} (\sqrt{z}) = \frac{\arctan(-1/3) + \pi + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt[4]{10} \left( \text{cis} \left( \arctan \left( \frac{-1}{3} \right) + \pi \right) \right)
\]

\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt[4]{10} \left( \text{cis} \left( \arctan \left( \frac{-1}{3} + 3\pi \right) \right) \right)
\]
59. Find all square roots of \( z = 2 + 7i \).

**Solution.** We first convert to polar coordinates. In Quadrant 1, the polar form of \( z \) is

\[
z = \sqrt{53} \left( \cos \left( \arctan\left(\frac{7}{2}\right) \right) + i \sin \left( \arctan\left(\frac{7}{2}\right) \right) \right)
\]

The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg \left( \sqrt{z} \right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{\sqrt{53}^2} = \sqrt{53}, \quad \arg \left( \sqrt{z} \right) = \frac{\arctan(7/2) + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt{53} \left( \text{cis} \left( \frac{\arctan(7/2)}{2} \right) \right)
\]

\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt{53} \left( \text{cis} \left( \frac{\arctan(7/2) + 2\pi}{2} \right) \right)
\]

60. Find all square roots of \( z = 2 + 3i \).

**Solution.** We first convert to polar coordinates. In Quadrant 1, the polar form of \( z \) is

\[
z = \sqrt{13} \left( \cos \left( \arctan\left(\frac{3}{2}\right) \right) + i \sin \left( \arctan\left(\frac{3}{2}\right) \right) \right)
\]

The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg \left( \sqrt{z} \right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{\sqrt{13}^2} = \sqrt{13}, \quad \arg \left( \sqrt{z} \right) = \frac{\arctan(3/2) + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt{13} \left( \text{cis} \left( \frac{\arctan(3/2)}{2} \right) \right)
\]

\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt{13} \left( \text{cis} \left( \frac{\arctan(3/2) + 2\pi}{2} \right) \right)
\]
61. Find all square roots of \( z = -1 + 5i \).

**Solution.** We first convert to polar coordinates. In Quadrant 2, the polar form of \( z \) is
\[
z = \sqrt{26} \left( \cos \left( \arctan \left( \frac{-5}{2} \right) \right) + i \sin \left( \arctan \left( \frac{-5}{2} \right) \right) \right)
\]
Remember that since arctangent is always in quadrants 1 or 4, we need a shift of \( \pm \pi \) to get the argument for \( z \). The formula for the modulus and argument of the square roots are
\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]
So for this case,
\[
|\sqrt{z}| = \sqrt{26}, \quad \arg(\sqrt{z}) = \frac{\arctan(-5) + \pi + 2k\pi}{2}, \quad k = 0, 1.
\]
Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become
\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt{26} \left( \cos \left( \frac{\arctan(-5) + \pi}{2} \right) + i \sin \left( \frac{\arctan(-5) + \pi}{2} \right) \right)
\]
\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt{26} \left( \cos \left( \frac{\arctan(-5) + 3\pi}{2} \right) + i \sin \left( \frac{\arctan(-5) + 3\pi}{2} \right) \right)
\]

62. Find all square roots of \( z = 2 - 7i \).

**Solution.** We first convert to polar coordinates. In Quadrant 4, the polar form of \( z \) is
\[
z = \sqrt{53} \left( \cos \left( \arctan \left( \frac{-7}{2} \right) \right) + i \sin \left( \arctan \left( \frac{-7}{2} \right) \right) \right)
\]
The formula for the modulus and argument of the square roots are
\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]
So for this case,
\[
|\sqrt{z}| = \sqrt{53}, \quad \arg(\sqrt{z}) = \frac{\arctan(-7/2) + 2k\pi}{2}, \quad k = 0, 1.
\]
Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become
\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt[4]{53} \left( \cos \left( \frac{\arctan(-7/2)}{2} \right) + i \sin \left( \frac{\arctan(-7/2)}{2} \right) \right)
\]
\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt[4]{53} \left( \cos \left( \frac{\arctan(-7/2) + 2\pi}{2} \right) + i \sin \left( \frac{\arctan(-7/2) + 2\pi}{2} \right) \right)
\]
63. Find all square roots of $z = 1 - 6i$.

**Solution.** We first convert to polar coordinates. In Quadrant 4, the polar form of $z$ is

$$z = \sqrt{37} \left( \cos \left( \arctan(-6) \right) + i \sin \left( \arctan(-6) \right) \right)$$

The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \quad \text{arg} \left( \sqrt{z} \right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$ 

So for this case,

$$|\sqrt{z}| = \sqrt{37} = \sqrt[4]{37}, \quad \text{arg} \left( \sqrt{z} \right) = \frac{\arctan(-6) + 2k\pi}{2}, \quad k = 0, 1.$$ 

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \left( \sqrt[4]{37} \right) \left( \text{cis} \left( \frac{\arctan(-6)}{2} \right) \right)$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \left( \sqrt[4]{37} \right) \left( \text{cis} \left( \frac{\arctan(-6) + 2\pi}{2} \right) \right)$$

64. Find all square roots of $z = -2 + 7i$.

**Solution.** We first convert to polar coordinates. In Quadrant 2, the polar form of $z$ is

$$z = \sqrt{53} \left( \text{cis} \left( \arctan(-7/2) + \pi \right) \right)$$

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of $\pm\pi$ to get the argument for $z$. The formula for the modulus and argument of the square roots are

$$|\sqrt{z}| = \sqrt{|z|}, \quad \text{arg} \left( \sqrt{z} \right) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.$$ 

So for this case,

$$|\sqrt{z}| = \sqrt{53} = \sqrt[4]{53}, \quad \text{arg} \left( \sqrt{z} \right) = \frac{\arctan(-7/2) + \pi + 2k\pi}{2}, \quad k = 0, 1.$$ 

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

$$k = 0 \quad \Rightarrow \quad \sqrt{z} = \left( \sqrt[4]{53} \right) \left( \text{cis} \left( \frac{\arctan(-7/2) + \pi}{2} \right) \right)$$

$$k = 1 \quad \Rightarrow \quad \sqrt{z} = \left( \sqrt[4]{53} \right) \left( \text{cis} \left( \frac{\arctan(-7/2) + 3\pi}{2} \right) \right)$$
65. Find all square roots of \( z = -2 - 7i \).

**Solution.** We first convert to polar coordinates. In Quadrant 3, the polar form of \( z \) is

\[
z = \sqrt{53} (\text{cis} (\arctan(7/2) + \pi))
\]

Remember that since arctangent is always in quadrants 1 or 4, we need a shift of \( \pm \pi \) to get the argument for \( z \). The formula for the modulus and argument of the square roots are

\[
|\sqrt{z}| = \sqrt{|z|}, \quad \arg(\sqrt{z}) = \frac{\arg(z) + 2k\pi}{2}, \quad k = 0, 1.
\]

So for this case,

\[
|\sqrt{z}| = \sqrt{\sqrt{53}} = \sqrt{53}, \quad \arg(\sqrt{z}) = \frac{\arctan(7/2) + \pi + 2k\pi}{2}, \quad k = 0, 1.
\]

Since we cannot evaluate the arctangent here, no further reduction is possible or required. Our two values become

\[
k = 0 \quad \Rightarrow \quad \sqrt{z} = \sqrt[4]{53} \left( \text{cis} \left( \frac{\arctan(7/2) + \pi}{2} \right) \right)
\]

\[
k = 1 \quad \Rightarrow \quad \sqrt{z} = \sqrt[4]{53} \left( \text{cis} \left( \frac{\arctan(7/2) + 3\pi}{2} \right) \right)
\]